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TABLE OF CONTENTS

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Mathematical Reviews

Vol. 10, No. 3 MARCH, 1949 Pages 173-228

HISTORY

***Struik, Dirk J. A Concise History of Mathematics.** Dover Publications, Inc., New York, N. Y., 1948. Vol. 1, pp. i–xviii+1–124; Vol. 2, pp. 125–299. $3.00.

A very condensed survey of the history of mathematics covering the whole development from the earliest times up to 1900. The conciseness of the work imposed the necessity of a strict limitation to the main currents of thought. This, however, did not prevent the author from paying due attention to what always has been his favourite topic, viz., the relations of mathematics to the general cultural and sociological background of the period. In this way the two trends which can be distinguished in Renaissance mathematics, the arithmetical-algebraic and what may be called the fluctional one, are related, respectively, to the commercial and to the engineering interests of these centuries. In treating of Greek mathematics, that most important, but very imperfectly known, phase of the development of mathematics, the author takes care to distinguish between established facts, plausible theories, wild hypotheses and traditional ideas. The exposition of 19th century mathematics is based more on persons and schools than on subjects. The work is richly illustrated. *E. J. Dijksterhuis* (Oisterwijk).

van der Waerden, B. L. Babylonian astronomy. II. The thirty-six stars. J. Near Eastern Studies **8**, 6–26 (1949).

This article gives a summary of the development of Babylonian astronomy in its earlier phases, i.e., from the 18th to the 5th century B.C. The origin of celestial coordinates is one of the main problems discussed. [Part I of this series concerns Old-Babylonian chronology and appeared in Ex Oriente Lux No. 10 (1948).] *O. Neugebauer.*

Mogenet, Joseph. La traduction latine par Gérard de Crémone du traité de la sphère en mouvement d'Autolycus. Arch. Internat. Hist. Sci. **28**, 139–164 (1948).

Autolycus, who lived shortly before Euclid, is the earliest Greek mathematician whose writings have been preserved. This fact is due to their having been included in a group of treatises used as introduction to the Almagest. One of these small works is the "Rotating Sphere," a qualitative spherical astronomy. The Greek text was edited in 1885 by Hultsch. The present paper contains the edition of a Latin version, obviously a translation from the Arabic by Gerhard of Cremona. The edition is based on four mss., the best of which is Par. lat. 9335 (saec. XIV). A short introduction by the editor precedes the Latin text. A superficial comparison between the Greek and the Latin version shows rather wide divergences in details, most conspicuous in theorem II where a second version from "aliis scriptis" is quoted. The editor is inclined to trace this back to Greek originals, no longer available to us. It would be interesting to know how far the figures given in the present edition reflect the situation in the mss. *O. Neugebauer.*

***Proclus de Lycie. Les Commentaires sur le Premier Livre des Éléments d'Euclide. Traduits pour la Première Fois du Grec en Français avec une Introduction et des Notes par Paul Ver Eecke.** Collection de Travaux de l'Académie Internationale d'Histoire des Sciences, No. 1. Desclée de Brouwer et Cie., Bruges, 1948. xxiv+372 pp.

Proclus's commentary on the first book of Euclid's "Elements," which is well known to experts on the history of Greek mathematics or on neoplatonist philosophy, may also interest less specialized readers. It has a long preface in two parts, the first mainly philosophical (and often obscure), the second mainly historical. This preface, together with the comments on the definitions, axioms and postulates of Euclid, occupies about one half of the work. The second half contains comments on the 48 propositions of the first book of the Elements with long, and sometimes remarkable, digressions. Thus the commentary on the first proposition contains remarks on Euclid's manner of exposition and, incidentally, on the methods of mathematical invention. Previously the work was scarcely accessible without a knowledge of Greek. A good Latin translation [Barocius, 1560] is extremely rare and an English translation [T. Taylor, 1792] rather bad. Ver Eecke's text seems very careful and as readable as the translation of an often obscure original can be. It has many useful footnotes. Some of these compare the French translation with the Greek original, others contain references or historical remarks. An introduction brings data on the life and works of Proclus, and on previous editions and translations of the Commentary. [Cf. the following review.] *G. Pólya.*

Proclus Diadochus. Kommentar zum ersten Buch von Euklids "Elementen." Aus dem Griechischen ins Deutsche übertragen und mit textkritischen Anmerkungen versehen von Leander Schönberger. Eingeleitet, mit Kommentaren und bibliographischen Nachweisen versehen und in der Gesamtedition besorgt von Max Steck. Deutsche Akademie der Naturforscher, Halle (Saale) ~~(Salle)~~, 1945. xxiv+592 pp. (4 plates)

The introduction [33 pp.] contains many words which fortunately have no English equivalent, e.g., "deutscher Geistraum," "Geistschau," "in- und ausstrahlen," etc. By means of this "denkanschauend" method Proclus is made a founder of the German Idealismus for which Cusanus, Copernicus, Kepler, Hegel, Gauss (!) and many others are quoted. On the other hand, Proclus is considered as the culmination of Greek mathematics. The author here follows Speiser with whom he shares the tendency to consider the last phase of Greek metaphysics as representative of Greek mathematics. The subsequent commentary on Proclus shows the same contempt for the chronological element of history. There is hardly a combination of any pair of famous names missing, however great their distance may be. Also

the mathematical commentary is far from conventional. One statement [p. 109] may suffice as a specimen. "Function" (or "curve") cannot be explained because of being a "höchster Gattungsbegriff." This commentary by Steck ends at p. 152 and finds a continuation in notes from p. 469 to the end. The rest of the work contains the translation of the Greek text, edited by Friedlein in 1873. The translation by Schönberger is indeed a very careful rendering of an extremely difficult text which has often been misunderstood by its editors. Thus many notes are found in the present translation suggesting more or less obvious modifications in Friedlein's edition. It is interesting to compare this translation with Ver Eecke's rendering [cf. the preceding review]. Frequently both scholars reach the same conclusion. There also exist, however, many divergencies. I mention only a new interpretation of Euclid's definition of the straight line where Schönberger translates "Eine Linie ist gerade, wenn sie gleich ist dem Abstand ihrer Endpunkte" as compared with the more traditional and certainly more literal ". . . également placée entre ses points." It is obvious that all future work with Proclus's Commentary will have to take Schönberger's translation into serious consideration. *O. Neugebauer.*

√ ✶**Sanchez Perez, Jose Augusto. La Aritmética en Grecia. [Arithmetic in Greece].** Consejo Superior de Investigaciones Cientificas, Madrid, 1946. 260 pp.

Cet ouvrage fait suite à un livre antérieur du même auteur [La Aritmética en Babilonia y Egipto, Madrid, 1943; ces Rev. 6, 253]. L'exposition systématique de la logistique et de l'arithmétique grecques, qui en forme l'objet principal, est précédée par trois chapitres préliminaires: (1) notices historiques et géographiques sur le monde ancien; (2) notices biographiques sur les créateurs de l'arithmétique grecque; (3) résumé du contenu des ouvrages classiques et fondamentaux dans le domaine de l'arithmétique, à savoir les livres 1, 2 et 5–9 d'Euclide, l'Introduction arithmétique de Nicomaque de Gerase et l'Arithmétique et le livre sur les nombres polygonaux de Diophante. *E. J. Dijksterhuis.*

Beaujouan, Guy. Étude paléographique sur la "rotation" des chiffres et l'emploi des apices du X^e au XII^e siècle. Rev. Hist. Sci. Appl. 1, 301–313 (1948).

This is a study in the development of the Arabic numerals and the abacus. One important thesis, which seems well founded in the palaeographical material, consists in the explanation of various forms as the result of a rotation of the original signs through various angles, in part caused by the change of direction of writing. *O. Neugebauer.*

✶**Kropp, Gerhard. Beiträge zur Philosophie, Pädagogik und Geschichte der Mathematik. Mit einem Anhang: Die mathematikgeschichtliche Forschung. Geometrische Integrationsmethoden bei Lalouvère.** Fr. K. Koetschau Verlag, Berlin, 1948. 103 pp.

Probably the only section which is expected to be of interest outside Germany is the appendix on Lalouvère [1600–1664] whose methods of integration are described here, as an excerpt from the author's thesis on Lalouvère's "Quadraturi circuli" [Toulouse, 1651]. *O. Neugebauer.*

Neugebauer, Otto. The astronomical origin of the theory of conic sections. Proc. Amer. Philos. Soc. 92, 136–138 (1948).

Scott, J. F. Mathematics through the eighteenth century. Philos. Mag., Commemoration Number, 67–91 (1948).

Greenwood, Thomas. Origines de la géométrie analytique. Rev. Trimest. Canad. 34, 166–179 (1948).

DeVries, H. How analytic geometry became a science. Scripta Math. 14, 5–15 (1948).

Palamà, Giuseppe. Similitudine dei triangoli ed uguaglianza dei triangoli e dei triedri. Boll. Un. Mat. Ital. (3) 3, 49–66 (1948).
Historical article.

Câmpan, Florica. The golden section. Revista Ştiinţifică "V. Adamachi" 33, 225–231 (1947). (Romanian)

Vera, Francisco. Les mathématiques à l'école des traducteurs de Tolède. Ann. Soc. Polon. Math. 21, 94–98 (1948).

Conte, Luigi. Giovanni Bernoulli e la sfida di Brook Taylor. Arch. Internat. Hist. Sci. 27, 611–622 (1948).

Ionescu, D. V. Obituary: N. Abramescu. Mathematica, Timişoara 23, 139–146 (1948).

Obituary: Stefan Banach. Uspehi Matem. Nauk (N.S.) 1, no. 3–4(13–14), 13–16 (1946). (Russian)

Obituary: Stefan Banach. Colloquium Math. 1, 68–73 (1 plate) (1948).

Steinhaus, H. Souvenir de Stefan Banach. Colloquium Math. 1, 74–80 (1948).

Orlicz, W. Sur l'oeuvre scientifique de Stefan Banach. I. Théorie des opérations et théorie des séries orthogonales. Colloquium Math. 1, 81–92 (1948).

Marczewski, E. Sur l'oeuvre scientifique de Stefan Banach. II. Théorie des fonctions réelles et théorie de la mesure. Colloquium Math. 1, 93–102 (1948).

Bouligand, G. A une étape décisive de l'algèbre: L'oeuvre scientifique et l'oeuvre didactique d'Étienne Bezout. Rev. Gén. Sci. Pures Appl. N.S. 55, 121–123 (1948).

Carruccio, Ettore. Obituary: Ettore Bortolotti. Period. Mat. (4) 26, 1–13 (1948).

Čebotarëv, N. G. Mathematical autobiography. Uspehi Matem. Nauk (N.S.) 3, no. 4(26), 3–66 (1948). (Russian)

Delone, B. Obituary: Nikolaĭ Grigor'evič Čebotarëv, 1894–1947. Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 337–340 (1948). (Russian)

Steklov, V. A. Theory and practice in Čebyšev's research. Uspehi Matem. Nauk (N.S.) 1, no. 2(12), 4–11 (1946). (Russian)
A lecture given in 1921.

Narciso Pérez. Fr. Feijóo and the natural sciences. A chapter in the history of Spanish science. Revista Acad. Ci. Madrid 41, 119–173, 287–337, 469–513, 599–643 (1947). (Spanish)
Benito Jeronimo Feijóo Fajarda lived 1676–1764.

Itard, Jean. Fermat, précurseur du calcul différentiel. Arch. Internat. Hist. Sci. **27**, 589–610 (1948).

Dupuy, P. The life of Evariste Galois. Translated by F. J. Duarte. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **10**, 219–299 (4 plates) (1947). (Spanish)
Translated from Ann. Sci. École Norm. Sup. (3) **13**, 197–266 (1896).

Revue d'Histoire des Science et de leurs

Taton, René. Les relations d'Évariste Galois avec les mathématiciens de son temps. Rev. Hist. Sci. Appl. **1**, 114–130 (1947). Applications

This paper contains an unpublished preface of Galois to a planned work "Deux mémoires d'Analyse pure" where he not only expresses his dislike for the members of the Academy but also indicates his views about the future development of mathematics, much of which has taken place in accordance with his predictions.
O. Neugebauer (Providence, R. I.).

Obituary: Nil Aleksandrovič Glagolev (1888–1945). Uspehi Matem. Nauk (N.S.) **1**, no. 2(12), 43–47 (1946). (Russian)

Milne, E. A. Obituary: Godfrey Harold Hardy. Monthly Not. Roy. Astr. Soc. **108**, 44–46 (1948).

Weyl, Hermann. David Hilbert and his mathematical work. Bol. Soc. Mat. São Paulo **1**, 76–104 (1946); **2**, 37–60 (1947). (Portuguese)
Translated from Bull. Amer. Math. Soc. **50**, 612–654 (1944); these Rev. **6**, 142.

Loria, Gino. La vita scientifica di Cristiano Huygens quale si desume dal suo carteggio. Pont. Acad. Sci. Comment. **6**, 1079–1136 (1942).

Obituary: Aleksandr Antipovič Kulakov (1898–1946). Uspehi Matem. Nauk (N.S.) **2**, no. 2(18), 185–187 (1947). (Russian)

Smirnov, V. I. The scientific work of Aleksei Nikolaevič Krylov. Uspehi Matem. Nauk (N.S.) **1**, no. 3–4(13–14), 3–12 (1946). (Russian)

Yuškevič, A. P. Leibniz and the foundations of the infinitesimal calculus. Uspehi Matem. Nauk (N.S.) **3**, no. 1(23), 150–164 (1948). (Russian)

Leibniz, G. W. Selections from the mathematical works of Leibniz, chosen and translated by A. P. Yuškevič. Uspehi Matem. Nauk (N.S.) **3**, no. 1(23), 165–204 (1 plate) (1948). (Russian)

Kuratowski, Casimir. Stefan Mazurkiewicz et son oeuvre scientifique. Fund. Math. **34**, 316–331 (1947).

Dugas, René. Le troisième centenaire de Newton. Revue Sci. **86**, 111–114 (1948).

Kuznecov, B. G. The teaching of Newton on relativity and absolute motion. Izvestiya Akad. Nauk SSSR. Ser. Istor. Filos. **5**, 149–166 (1948). (Russian)

Bortolotti, Ettore. Il carteggio matematico di Paolo Ruffini. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **3**, 215–224 (1947).

Kolmogorov, A. N. Obituary: Evgenii Evgenievič Sluckii. Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 143–151 (1 plate) (1948). (Russian)

Smirnov, N. Obituary: Evgenii Evgen'evič Sluckii, 1880–1948. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 417–420 (1948). (Russian)

The usual transliteration is Slutsky.

Smirnov, V. I. Vladimir Andreevič Steklov (on the occasion of the 20th anniversary of his death). Uspehi Matem. Nauk (N.S.) **1**, no. 3–4(13–14), 17–22 (1946). (Russian)

Gyunter, N. M. The work of V. A. Steklov in mathematical physics. Uspehi Matem. Nauk (N.S.) **1**, no. 3–4(13–14), 23–43 (1946). (Russian)

Sansone, G. L'opera scientifica di Leonida Tonelli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 594–624 (1948).

Somigliana, Carlo. Obituary: Vito Volterra. Pont. Acad. Sci. Acta **6**, 57–85 (1942).

Hammerschmidt, William W. Obituary: Alfred North Whitehead. Scripta Math. **14**, 17–23 (1 plate) (1948).

FOUNDATIONS

Mostowski, Andrzej. On a set of integers not definable by means of one-quantifier predicates. Ann. Soc. Polon. Math. **21**, 114–119 (1948).

This is a sequel to a previous paper of the author on definable sets of integers [Fund. Math. **34**, 81–112 (1947); these Rev. **9**, 129, where the necessary notations are explained]. It is now proved that $R_1^{(n)}$, the smallest finitely additively field of sets containing $P_1^{(n)}$ and $Q_1^{(n)}$, is a proper subset of $P_2^{(n)} \cap Q_2^{(n)}$. (That it is a subset follows from 2·18 and 2·31 of the previous paper.) The significance of this result is that there exist predicates of integers which have definitions of the form $(Ex)(y)\phi(x, y, \mathfrak{n})$ and $(y)(Ex)\psi(x, y, \mathfrak{n})$, where $\mathfrak{n}$ is an n-tuple of integers and x and y are integer-variables, and ϕ and ψ are general-recursive; but which have no definition compounded by means of the algebra of logic out of expressions $(x)\phi_i(x, \mathfrak{n})$ and $(Ex)\psi_j(x, \mathfrak{n})$, where the ϕ_i and ψ_j are general-recursive. Such a predicate is constructed.
M. H. A. Newman (Manchester).

Jaśkowski, Stanisław. Un calcul des propositions pour les systèmes déductifs contradictoires. Studia Soc. Sci. Torunensis. Sect. A. **1**, 57–77 (1948). (Polish. French summary)

Call a system inconsistent if some A and non-A are both theorems, and call it overcompleted if every formula is a theorem. These notions are equivalent in classical logic, for we have '$CpCNpq$' (called by the author the law of overcompletion), but they are not equivalent in systems due to Kolmogoroff, Lewis (among others), or in many-valued logics. The author is looking for a new system of sentential calculus which will be intuitive and powerful, but which when applied to inconsistent systems will not always lead

to overcompleteness. He constructs a system D_2 of two-valued sentential calculus satisfying these requirements in the following way. First take functional calculus of the first order where 'x' is the only variable bound by quantifiers and where all functions are monadic (i.e., taking only one argument). This system has a decision procedure suggested by Behmann [Hilbert and Ackermann, Grundzüge der theoretischen Logic, 1st ed., Springer, Berlin, 1928, pp. 77–78], and by interpretation in it we form a modal system M_2 similar to Lewis's S5, writing '$p(x)$', '$q(x)$', $\cdots$ instead of 'p', 'q', $\cdots$, 'for all x' instead of 'necessary' and 'for some x' instead of 'possible.' If an expression becomes a theorem in functional calculus it is a theorem of M_2. Then D_2 is formed by interpretation in M_2. Jaśkowski defines so-called discursive implication as $Cd\,p\,q = C\,\mathrm{Pos}\,p\,q$ and discursive equivalence as $Ed\,p\,q = KC\,\mathrm{Pos}\,p\,q\,C\,\mathrm{Pos}\,q\,\mathrm{Pos}\,p$. An expression is a theorem of D_2 if (1) it contains only sentential variables and at most the following functors: Cd, ED, A, K, N, and if (2) preceded by 'Pos' it forms a theorem of M_2. Thus the assertion of a theorem in D_2 amounts only to stating a possibility. The author accepts modus ponens for 'Cd,' viz.: $Cd\,p\,q$, $p \rightarrow q$. The system D_2 is very powerful, but, e.g., '$Cd\,p\,Cd\,q\,K\,p\,q$,' '$Cd\,N\,Cd\,ppq$,' the law of overcompleteness and several of its instances, and all forms of transposition are not valid in it. The usual derivations of antinomies fail. Altogether the system seems to be very close to the intuition of dialecticians.

H. Hiż (Cambridge, Mass.).

Smullyan, Arthur Francis. Modality and description. J. Symbolic Logic **13**, 31–37 (1948).

The author discusses a form of the antinomy of the name-relation [R. Carnap, Meaning and Necessity, University of Chicago Press, 1947, Chaps. 3, 5; these Rev. **8**, 430] as illustrated by the following example due to W. V. Quine [J. Philos. **40**, 113–127 (1943)]. (A) It is logically necessary that 9 is less than 10; (B) 9 = the number of the planets; therefore (C) it is logically necessary that the number of the planets is less than 10. He shows that, when due attention is given to the scope of the descriptions used, it is possible by Russell's method of contextual definition of descriptions [Mind (N.S.) **14**, 479–493 (1905); Principia Mathematica, v. 1, *14] to eliminate this difficulty without restricting the use of modal operators prefixed to the sentences of the system or violating Leibniz' principle, that if x and y are identical, then y has every property of x. Given two premisses $N(Fy)$ and $y = (\iota x)(\varphi x)$ (where 'N' is the modal sign for logical necessity), it is possible to derive within the system of Principia Mathematica the conclusion $[(\iota x)(\varphi x)].N(F(\iota x)(\varphi x))$, but not $N([(\iota x)(\varphi x)].F(\iota x)(\varphi x))$. This same analysis applies to modal operators used in combination with class-abstracts provided these also are not treated as names but as incomplete symbols.

J. J. de Iongh (Amsterdam).

Tarski, Alfred. A problem concerning the notion of definability. J. Symbolic Logic **13**, 107–111 (1948).

This paper is a sequel to Tarski's earlier studies on definability [cf. Fund. Math. **17**, 210–239 (1931); Erkenntnis **5**, 80–100 (1935)], which are closely related to his semantical truth definition and to the Gödel theorem. It is generally believed that, by means of an argument analogous to the one which leads to the Richard paradox, the notion of definability as applied to entities discussed in a formal system can easily be shown not to be itself definable in that system. The author shows, however, that the situation is more complicated than that. Consider a formal system $\mathfrak{S}$, analogous to that of Principia Mathematica, but based upon the simple theory of types. Variables of the zeroth order are interpreted as representing natural numbers; variables of the first order represent sets of natural numbers, i.e., real numbers, etc. We must maintain a distinction between two different interpretations of the notion of definability, which, however, are equivalent. (1) Consider an element a of the nth order ($n = 0, 1, 2, \cdots$). It will be considered as definable in $\mathfrak{S}$ if and only if there is in $\mathfrak{S}$ a formula ϕ which contains a certain variable of the nth order as the only free variable, which is satisfied by a and only by a; this formula ϕ is said to define the element a. (2) Consider an element a of the nth order ($n = 1, 2, \cdots$). Such an element may be considered as a set of elements of the $(n-1)$st order and therefore it may be said to be defined by a formula ψ which contains a certain variable of the $(n-1)$st order as its only free variable and which is satisfied by the elements of a and only by these elements.

Now we consider the set D_n of the elements of the nth order which are definable in $\mathfrak{S}$, and we ask whether this set (which obviously is an element of the $(n+1)$st order) is definable in $\mathfrak{S}$, i.e., whether D_n is a member of D_{n+1}. Tarski distinguishes three cases. (1) $n = 0$. In this case, the solution is trivially positive, the elements of zeroth order, i.e., the natural numbers, being definable in $\mathfrak{S}$. (2) $n = 2$. In this case the solution is negative, on account of an argument derived from the analysis of the Richard paradox. This argument reduces essentially to the fact that we can construct a function f which correlates, with every denumerable set a of sets of real numbers, a set $f(a)$ of real numbers which does not belong to a. (3) $n = 1$. In this case the argument fails, as we do not know whether a function analogous to the function f can be defined for denumerable sets of real numbers. Thus the problem remains open whether the notion of a real number definable in a formal system $\mathfrak{S}$ (in which the theory of real numbers can adequately be formalized) is itself definable in $\mathfrak{S}$. Tarski observes, however, that this problem is closely related to K. Gödel's axiom of constructibility [The Consistency of the Continuum Hypothesis, Princeton University Press, 1940; these Rev. **2**, 66], which makes it seem very unlikely that an affirmative solution of the problem is possible.

E. W. Beth (Amsterdam).

***Kraft, Victor. Mathematik, Logik und Erfahrung.** Springer-Verlag, Wien, 1947. vii+129 pp. $2.40.

This book is concerned with the conditions that mathematics and logic be applicable to experience. It consists of three parts.

(I) Die empirische Geltung der Mathematik. Principal conclusions: (a) "Eine Klassenmenge, eine Menge, deren Elemente bloss generell durch ein Kriterium bestimmt sind, kann als solche nicht zahlenmässig bestimmt werden; denn die Anzahl ihrer Elemente ist damit noch ganz ungewiss. Die Klassenmenge muss erst in eine Aufzählungsmenge verwandelt werden, um abzählbar zu sein. . . . Die Bedingung für die Anwendbarkeit der natürlichen Zahlen ist daher nur die, dass in der Erfahrung eine Mehrheit von Einzelnem gegeben sein muss." (b) "Eine Geometrie an und für sich überhaupt keine Beziehung zur Erfahrung hat und gar nicht unmittelbar für sie gelten kann. . . . Deshalb muss erst eine Zuordnung von empirischen Objekten zu den Elementen und Beziehungen einer reinen Geometrie

hergestellt werden." The author describes the problems of setting up such a correspondence.

(II) Widerlegung des Konventionalismus. Summary: "Übereinstimmung der Gesetze mit der Erfahrung d.i. mit Wahrnehmungen ist darum auch für den Konventionalismus eine unerlässliche Bedingung. Aber diese Übereinstimmung bildet für ihn nicht mehr ein Kriterium für wahr oder falsch. Denn sie lässt sich nach ihm unter allen Umständen, für jede beliebige Festsetzung von Gesetzen herstellen. . . . In [certain stated] sachlichen Bedingungen liegt die Rechtfertigung nicht nur, sondern die unumgängliche Notwendigkeit des Empirismus, der die Antithese des Konventionalismus ist, und dieser wird eben dadurch widerlegt."

(III) Die Anwendungsbedingungen der Logik. Conclusion: "Die empirische Anwendung der Logik ist somit an eine Bedingung gebunden: an die Herstellbarkeit von Ordnung, von Gesetzmässigkeit; und die Erfüllung oder Nicht-Erfüllung dieser Bedingung ergibt einen Unterschied in der Art einer Mannigfaltigkeit im Sinn von rational und irrational. Damit ist dann eben gemeint: der Anwendbarkeit der Logik gemäss oder nicht gemäss."

C. C. Torrance (Annapolis, Md.).

ALGEBRA

Schulz, Günther. Zwei Hilfssätze aus der Kombinatorik. Z. Angew. Math. Mech. **28**, 274–275 (1948).

The two propositions proved are equivalent to (i) the number of ways of selecting k sets, each consisting of l consecutive elements, from elements 1 to n, is $S(n, k)=\binom{n-kl+k}{k}$; (ii) when consecutivity is cyclic, the number of ways is $lS(n-l, k)+S(n-1, k)$. Applications to runs, counters and trunking problems are contemplated. *J. Riordan.*

Lévy, Paul. Étude d'une classe de permutations. C. R. Acad. Sci. Paris **227**, 422–423 (1948).

Permutations of n elements defined by (a) $y=2x-1$, $2x-1\leqq n$; (b) $y=2(n+1-x)$, $2x-1>n$, are examined for type of cycle structure, the type of a cycle being the sequence of operations (a) and (b) which it involves; e.g., $n=7$, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 5 & 7 & 8 & 6 & 4 & 2 \end{pmatrix} = (1)\ (2358)\ (47)\ (6)$$

is of type (a) (aabb) (ab) (b). Twelve propositions are stated, proof being promised elsewhere, chief interest being on the first appearance of primitive types, which correspond to cycles whose order is not repeated in the permutation. Also the modifications in these propositions are given for n odd and equal to $2q-1$ and the operations (a) $y=2x-1$, $x\leqq q$; (c) $y=2(x-q)$, $x>q$. [Reviewer's note: the permutations first considered above are solutions of what may be called the hostess problem, which is also a telephone cable-splicing problem and a linear graph problem [see Narasimha Murti, J. Indian Math. Soc. (N.S.) **4**, 39–43 (1940); Levi, ibid., 45–46 (1940); these Rev. **2**, 115].] *J. Riordan.*

Piza, P. A. Kummer numbers. Math. Mag. **21**, 257–260 (1948).

Two kinds of numbers appear: the first, denoted by ${}_nD_e$, are the differences of zero, that is, Δ^e0^n; the second are called Kummer numbers because of a connection noted by Ore [letter to the author] with Kummer polynomials associated with Fermat's last theorem, and of interest here because of their use in expressing multiple sums of powers. The second have a long and varied history going back to Laplace, a recent entry in which is a paper by Kaplansky and the reviewer [Duke Math. J. **13**, 259–268 (1946); these Rev. **7**, 508] which gives a brief indication of this history and further references. *J. Riordan* (New York, N. Y.).

Goodman, A. W. The number of terms in the expansion of an infinite determinant. Amer. Math. Monthly **55**, 419–420 (1948).

This is an alternative proof of a theorem of N. J. Lennes [Bull. Amer. Math. Soc. **18**, 22–24 (1911)] that the number of terms in the expansion of an infinite determinant has the power of the continuum. *C. C. MacDuffee.*

Hamburger, Hans Ludwig. Remarks on the reduction of a linear transformation to Jordan's normal form. J. London Math. Soc. **22** (1947), 173–179 (1948).

The author describes a reduction of a complex matrix to Jordan normal form which does not involve determinants and states that after some modifications the method can be used in Hilbert and Banach spaces. *C. C. MacDuffee.*

Grimshaw, M. E. On the matrix equation $(AX)^n=\|A\|I$. Proc. Cambridge Philos. Soc. **44**, 292–294 (1948).

If A is a normal matrix of order n, there is a unitary matrix X such that $(AX)^n=(XA)^n=|A|\cdot I$. If A is a circulant, X may be taken as diagonal. *C. C. MacDuffee.*

Wiegmann, N. A. Some analogs of the generalized principal axis transformation. Bull. Amer. Math. Soc. **54**, 905–908 (1948).

Two theorems are proved. (1) If $\{A_i\}$ is an arbitrary set of nonzero $r\times s$ matrices, there are unitary matrices U and V of orders $r\times r$ and $s\times s$, respectively, such that $UA_iV=D_i$, D_i diagonal and real, if and only if $A_iA_j^{ct}=A_jA_i^{ct}$ and $A_j^{ct}A_i=A_i^{ct}A_j$ for all i and j. (2) A necessary and sufficient condition that a set of $n\times n$ matrices $\{A_i\}$ be brought into diagonal forms by the same unitary U, V equivalence transformation, $UA_iV=D_i$, is that the products $A_iA_j^{ct}$ and $A_j^{ct}A_i$ be normal for all i, j and that $A_k(A_j^{ct}A_i)=(A_iA_j^{ct})A_k$ for all i, j and k. It is stated that a simple example shows the last condition in the second theorem cannot be omitted.

W. Givens (Knoxville, Tenn.).

Currie, J. C. Cassini ovals associated with a second order matrix. Amer. Math. Monthly **55**, 487–489 (1948).

If $A=(a_{ij})$, $i, j=1, 2, \cdots, n$, is a square matrix over the complex field, each characteristic root of A lies in the interior or on the boundary of at least one of the $n(n-1)/2$ Cassini ovals $|z-a_{kk}||z-a_{mm}|=P_kP_m$, where $P_k=\sum|a_{kj}|$, summed over $1\leqq j\leqq n$, $j\neq k$ [A. Brauer, Duke Math. J. **14**, 21–26 (1947); these Rev. **8**, 559]. If $n=2$ there is thus defined by A a unique Cassini oval. Let M be the set of all unitary transforms of A and let C be the set of Cassini ovals defined by the matrices of M. The field of values is the same for each of the matrices of M and is an ellipse. The author proves the theorem: the set of points covered by the interiors and the boundaries of the Cassini ovals of C consists of the interior and circumference of the director circle of the ellipse. *J. Williamson* (Flushing, N. Y.).

Todd, J. A. Combinants of a pencil of quadric surfaces. III. Proc. Cambridge Philos. Soc. **44**, 186–195 (1948).

The line complexes belonging to the irreducible system of two quaternary quadratic forms S and S' are sixteen in number [H. W. Turnbull, Proc. London Math. Soc. (2) **18**, 69–94 (1919)]. The author determines a set of combinants of the pencil $\lambda_0 S+\lambda_1 S'$, in which the sixteen coefficients of the power products of λ_0, λ_1 form a system equivalent to that of the sixteen complexes mentioned above. This set K consists of two forms independent of λ, two linear forms, two quadratic forms and one cubic form. He then obtains fundamental syzygies connecting the sixteen complexes as syzygies between combinantal forms. [Cf. the following review.] *J. Williamson* (Flushing, N. Y.).

Todd, J. A. Combinants of a pencil of quadric surfaces. IV. Proc. Cambridge Philos. Soc. **44**, 196–199 (1948).

In determining a complete system of combinantal line complexes of S and S' [see the preceding review] the author proceeds as in two previous papers [same Proc. **43**, 475–487, 488–490 (1947); these Rev. **9**, 170, 171]. He considers the five forms of the set K which involve λ explicitly and the quartic in λ whose coefficients are the five quadratic invariants of S and S'. Treating these six forms as arbitrary binary forms in λ he determines a complete system of invariants of the six forms. This system, when the coefficients are specialized to give the forms of K, contains the required system of combinantal line complexes. However, many of the members are reducible. The syzygies, obtained in the previous paper, are used to obtain from this system a complete system of combinantal line complexes of the two forms S and S' containing eighteen members.
J. Williamson (Flushing, N. Y.).

Lee, H. C. Sur le théorème de Hurwitz-Radon pour la composition des formes quadratiques. Comment. Math. Helv. **21**, 261–269 (1948).

Let $g(y_1, \cdots, y_n)$ and $h(z_1, \cdots, z_n)$ denote real positive-definite n-ary quadratic forms. A. Hurwitz obtained, in the case of the field of complex numbers, necessary and sufficient conditions for the existence of a bilinear transformation under which $(x_1^2+\cdots+x_p^2)g(y_1, \cdots, y_n)=h(z_1, \cdots, z_n)$ [Math. Ann. **88**, 1–25 (1923)]. J. Radon [Abh. Math. Sem. Hamburg. Univ. **1**, 1–14 (1921)] showed that the same conditions held, in the case of real forms, and that these conditions can be expressed as follows: $p\leqq 8\alpha+2^\beta$, where $4\alpha+\beta$ ($\beta=0, 1, 2$, or 3) denotes the exponent of the power of 2 dividing n. B. Eckmann [same Comment. **15**, 358–366 (1943); these Rev. **5**, 30, 225] gave a proof for the real case, based on a theory of group characters. Lee now gives another proof, for the real case, based on a representation of the Clifford algebra (due to Weyl and Brauer), and gives an actual construction of the solutions. For an allied problem with indefinite forms, see Lee [Proc. Nat. Acad. Sci. U. S. A. **33**, 379–381 (1947); these Rev. **9**, 324].
G. Pall (Chicago, Ill.).

Abstract Algebra

Cohn, Richard M. A note on the singular manifolds of a difference polynomial. Bull. Amer. Math. Soc. **54**, 917–922 (1948).

It is known that the manifold of a system of difference polynomials over a difference field of characteristic 0 is the union of a unique finite set of irreducible such manifolds none of which contains another; these irreducible manifolds are called the essential manifolds of the system. The author studies the essential manifolds of a single irreducible (i.e., nonfactorizable) difference polynomial A in unknowns $y_1, \cdots, y_n$. He proves that A has but two kinds of essential manifolds, ordinary ones and singular ones. An essential manifold M of A is singular if for every k ($1\leqq k\leqq n$) there exists a difference polynomial of lower effective order than A in y_k which vanishes on M, whereas M is ordinary if for no k does such a difference polynomial exist (effective order of A in y_k means order of highest transform of y_k present in A minus order of lowest such transform). The author shows further that, if the highest transform of y_k present in A is denoted by z_k, then ordinary and singular manifolds of A can be characterized as follows: if M is singular then for every k ($1\leqq k\leqq n$), $\partial A/\partial z_k$ vanishes on M; if M is ordinary then for no k does $\partial A/\partial z_k$ vanish. *E. R. Kolchin.*

Schwarz, Štefan. On generalization of the Jordan-Kronecker's "principle of reduction." Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1948**, no. 2, 1–27 (1948). (English. Czech summary)

Let $f(x)$, $g(x)$, and $h(x)$ be three irreducible polynomials over a field k. The author uses the Kronecker product of their root fields to deduce theorems (too complicated for brief summary) about the factorizations of one of the polynomials in the fields obtained by adjoining to k one root of each of the others. He has discussed the case of two polynomials in earlier papers [Časopis Pěst. Mat. Fys. **71**, 17–20 (1946); **72**, 61–64 (1947); these Rev. **8**, 500; **9**, 266].
G. Whaples (Bloomington, Ind.).

Krull, Wolfgang. Parameterspezialisierung in Polynomringen. Arch. Math. **1**, 56–64 (1948).

Let $K[x_1, \cdots, x_n]$ be a polynomial ring over an arbitrary field K, and let $\mathfrak{A}$ be an ideal in $K(u)[x_1, \cdots, x_n]$, where the groundfield K has been subjected to a simple transcendental extension. Substitute for u an element $a\epsilon K$ into the polynomials of $\mathfrak{A}$ whose denominators do not vanish at a, and designate by $\bar{\mathfrak{A}}$ the ideal generated by the resulting polynomials in $K[x]$. For ideals $\mathfrak{A}$, $\mathfrak{B}$ one verifies immediately that $\overline{\mathfrak{A}\cdot\mathfrak{B}}=\bar{\mathfrak{A}}\cdot\bar{\mathfrak{B}}$, $\overline{\mathfrak{A}+\mathfrak{B}}=\bar{\mathfrak{A}}+\bar{\mathfrak{B}}$ "almost always," i.e., but for at most a finite number of specializations. As for the quotient $\mathfrak{M}:\mathfrak{N}$, assume first that $\mathfrak{N}=(n(x))$ is principal (the general case will follow at once), and let $\mathfrak{M}=(m_1(x), \cdots, m_r(x))$. For any given integer N, let $p(x)$ be an element of $\mathfrak{M}:(n)$ of degree at most N. Then $p(x)\cdot n(x)=\sum m_i(x)\zeta_i(x)$, where, moreover, the degrees of the ζ_i can be bounded beforehand. This yields a system of linear equations of rank ρ in the coefficients of the ζ_i, and the vanishing of (certain) $(\rho+1)$-order determinants is necessary and sufficient for $p(x)$, considered as a specialization of the general polynomial of degree N, to belong to $\mathfrak{M}:(n)$. If the rank ρ is maintained upon specialization of u, one sees that these linear conditions on a polynomial (of degree not exceeding N) to belong to $\mathfrak{M}:\mathfrak{N}$ go over almost always into the linear conditions for a polynomial to belong to $\bar{\mathfrak{M}}:\bar{\mathfrak{N}}$. In particular, the linear conditions that a polynomial belong to $\mathfrak{A}$ go over, almost always, into the conditions that a polynomial belong to $\bar{\mathfrak{A}}$. (Place $\mathfrak{A}=\mathfrak{M}$, $\mathfrak{N}=(1)$!). Thus almost always $\overline{\mathfrak{M}:\mathfrak{N}}=\bar{\mathfrak{M}}:\bar{\mathfrak{N}}$. Similarly $\overline{\mathfrak{M}\cap\mathfrak{N}}=\bar{\mathfrak{M}}\cap\bar{\mathfrak{N}}$ almost always.

A prime ideal need not almost always specialize to a prime ideal, but an unmixed r-dimensional ideal specializes almost

always to an unmixed r-dimensional ideal. For the proof, Krull makes use of the "ground ideals" of a given ideal $\mathfrak{A}$ [E. Noether, Math. Ann. **90**, 229–261 (1923)]; in particular, the unmixed character of $\mathfrak{A}$ can be read off from the ground ideals. Moreover, G. Hermann [Math. Ann. **95**, 736–788 (1926)] has given a canonical algorithm for obtaining these ground ideals. The theorem now follows, almost trivially, from the fact that upon specialization this canonical algorithm specializes almost always into the canonical algorithm for $\bar{\mathfrak{A}}$.

Krull next considers the normal decomposition of $\mathfrak{A}$ and of $\bar{\mathfrak{A}}$ and thus brings the "specialization of parameters" problem near to a close. The questions still left open are to be discussed in a second note. *A. Seidenberg.*

Lesieur, Léonce. Anneaux réguliers, anneaux de matrices. J. Math. Pures Appl. (9) **27**, 205–253 (1948).

[Some of the results were announced in C. R. Acad. Sci. Paris **223**, 1083–1085 (1946); **224**, 321–323 (1947); these Rev. **8**, 249, 309]. A ring R is right regular if for any elements a, b there exists a two by two matrix P, not a left divisor of 0, such that $(a\ b)P=(c\ 0)$. If R has no divisors of 0, this coincides with Ore's definition: any two nonzero elements have a nonzero right common multiple. The principal theorem in chapter I asserts that the defining property may be extended to any rectangular matrix A, i.e., there exists a square matrix P, not a left divisor of 0, such that AP has zeros above the diagonal. The proof is similar to the usual reduction of matrices by elementary transformations.

Let us refer to right regularity as property 1. In chapters II and III four other properties are studied. (2) Every right divisor of 0 is a left divisor of 0. (3) Every nondivisor of 0 has an inverse. (4) We may write $(a\ b)P=(c\ 0)$, where P is a two by two matrix with an inverse. (5) The right annihilator of any element is a principal right ideal. It is shown that a ring with properties 1 and 2 can be embedded in one having 1 and 3. Also the following carry over from R to the ring of matrices over R: 1, 1 and 2, 1 and 3, 4, 4 and 5. Applications are made to linear equations, linear dependence, modules, and projective spaces.
I. Kaplansky (Princeton, N. J.).

Johnson, R. E. Equivalence rings. Duke Math. J. **15**, 787–793 (1948).

A ring is an equivalence ring if for any two of its elements a, b it is possible to find elements p, q such that either $a=pbq$ or $b=paq$. An example is the ring of linear transformations of a finite or infinite dimensional space over a division ring. A commutative integral domain is an equivalence ring if and only if it is a valuation ring. If a is an element of an equivalence ring, let F_a denote the set of elements x of the form paq, but such that a is not of the form $p'xq'$; the lattice formed by the F_a is studied, particularly for the ring of linear transformations mentioned above, and for commutative rings. These results are applied to the study of dense rings of linear transformations.
I. S. Cohen (Cambridge, Mass.).

Kaplansky, Irving. Topological rings. Bull. Amer. Math. Soc. **54**, 809–826 (1948).

This address is divided into three parts, topological division rings, locally compact rings and normed algebras. These items have, of course, important intersections, for instance normed division rings (in the real case) turn out to be identical with all finite division algebras over the real numbers which are, on the other hand, characterised as connected locally compact fields. A number of problems and conjectures are included and a bibliography of 83 items is given.
O. Todd-Taussky (London).

Barsotti, Iacopo. Ricerche sopra le algebre divisorie di tipo 1, e sopra le algebre divisorie non algebriche. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 1–30 (1948).

A division algebra is said to be of type 1 if every finite subset is contained in a finite-dimensional central subalgebra. Let F be a field over which the direct product of finite-dimensional division algebras splits unless the degrees are relatively prime. Then for any algebra A of type 1 over F there is a subalgebra D of countable order (a "characteristic" subalgebra) such that every countable subalgebra of A is isomorphic to a subalgebra of D. In order for uncountable algebras to exist with D as characteristic subalgebra, it is necessary and sufficient that D admit an isomorphism into a proper subalgebra of itself. If F is uncountable, it is shown that such isomorphisms can in fact exist, and hence that uncountable algebras of type 1 exist. A subsidiary result for finite-dimensional central division algebras is that they admit no proper subalgebras invariant under all automorphisms [this has also been proved by Cartan, Ann. Sci. École Norm. Sup. (3) **64**, 59–77 (1947); these Rev. **9**, 325].

In part II of the paper the author studies transcendental division algebras. If K is a division ring with automorphism σ, one can form the ring of polynomials in x with multiplication defined by $xa=a^{\sigma}x$, and one can embed the latter in a division ring B. Various results are given on the center C of B, of which the following is typical: if K is commutative and F the fixed field under σ, then C is F or $F(x^h)$ according as σ is of infinite order or of finite order h. The methods used are of a direct computational kind.
I. Kaplansky (Princeton, N. J.).

Barsotti, Iacopo. Elementi algebrici di algebre divisorie non algebriche. Ann. Scuola Norm. Super. Pisa (2) **14** (1945), 31–45 (1948).

The author continues the study begun in part II of the paper reviewed above. The methods are deeper and make use of the structure theory of finite algebras. In the previous notation, let now K be an algebra over F, and σ an F-automorphism. Principal attention is devoted to the elements of B algebraic over F. If K is of type 1, and no power of σ is inner, the only such elements are conjugates of elements of K. Under the weaker assumption that K is merely algebraic, the author shows that the only algebraic elements in the center of B are the σ-invariant elements of the center of K. In any nonalgebraic division algebra the totality of algebraic elements is a subring if and only if it is contained in the center. Otherwise the set of maximal algebraic subalgebras forms a groupoid in a natural way. Properties of this groupoid are studied. *I. Kaplansky.*

Barsotti, Iacopo. Sopra alcune proprietà delle sub-algebre normali di un'algebra di tipo 1. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 184–193 (1948).

Counter-examples are constructed to defeat two conjectures announced in the second last paper reviewed above. The conjectures are as follows. (1) If A is an algebra of type 1 and countable order, and B, B' are isomorphic subalgebras, then there exists an isomorphism of A into itself inducing the given isomorphism of B, B'. (2) If A is an

algebra of type 1 and countable order, and B is a proper central subalgebra, then the commutator of B is the base field if and only if B is isomorphic to A. *I. Kaplansky.*

Eilenberg, Samuel. Extensions of general algebras. Ann. Soc. Polon. Math. 21, 125–134 (1948).

Une algèbre A sur un corps commutatif F est un espace vectoriel sur F muni d'une application bilinéaire (multiplication) de $A\times A$ dans A. Un ensemble abstrait X définit comme suit une algèbre $\mathfrak{P}(X, F)$: l'espace vectoriel de $\mathfrak{P}(X, F)$ est somme directe de $\mathfrak{P}^n(X, F)$ (n entier non-négatif); $\mathfrak{P}^0(X, F)=F$; $\mathfrak{P}^n(X, F)$ a pour base l'ensemble $\mathfrak{M}^n(X)$ des "monômes d'ordre n" défini par récurrence sur n: $\mathfrak{M}^1(X)=X$; pour $n\geqq 2$, $\mathfrak{M}^n(X)$ se compose des couples ordonnés d'éléments x, y tels que $x\epsilon\mathfrak{M}^p$, $y\epsilon\mathfrak{M}^q$, $p+q=n$, $p\geqq 1$, $q\geqq 1$. La multiplication évidente des monômes définit une structure d'algèbre (graduée) sur $\mathfrak{P}(X, F)$: algèbre des "polynomes" (non associatifs) par rapport aux éléments de X, à coefficients dans F. Toute application λ de X dans une algèbre A se prolonge (d'une seule manière) en un homomorphisme $\bar\lambda$ de $\mathfrak{P}(X, F)$ dans A. On dit que A annule un polynome p si $\bar\lambda(p)=0$ pour toute application λ de X dans A. Exemple: une algèbre est "associative" si elle annule le polynome $(x_1x_2)x_3-x_1(x_2x_3)$.

Soient A et V deux algèbres sur F. Une extension de A par V est un système (E, α, β) formé d'une algèbre E, d'un isomorphisme α de V dans E, et d'un homomorphisme β de E sur A ayant pour noyau $\alpha(V)$. On suppose désormais que le produit de 2 éléments quelconques de V est nul; alors les applications de $E\times V$ dans V et de $V\times E$ dans V définies par la multiplication de E, définissent des applications bilinéaires φ et ψ de $A\times V$ dans V et $V\times A$ dans V. Réciproquement, un couple de telles applications bilinéaires φ et ψ définit une famille d'extensions de A par V, en correspondance biunivoque avec l'espace vectoriel, quotient de l'espace des applications bilinéaires f de $A\times A$ dans V, par le sous-espace des "cobords" d'applications linéaires h de A dans V; formule du cobord:

$$h(a_1, a_2)=\varphi(a_1, h(a_2))-h(a_1a_2)+\psi(h(a_1), a_2).$$

L'extension "triviale" définie par φ et ψ est celle qui correspond à $f=0$: E est somme directe de A et V, la multiplication entre éléments de la sous-algèbre A et de l'idéal V étant définie par φ et ψ.

Problème: A annulant un polynome p d'ordre $\geqq 2$, à quelles conditions doivent satisfaire φ, ψ et f pour que l'extension correspondante annule p? Réponse: il faut et il suffit que: (1) l'extension triviale définie par φ et ψ annule p; (2) on ait $f_\lambda(p)=0$ pour toute application λ de X dans A, où l'application linéaire f_λ de $\mathfrak{P}(X, F)$ dans V est définie (en fonction de f, λ, φ et ψ) par récurrence sur l'ordre n d'un polynome $q\epsilon\mathfrak{P}^n(X, F)$: $f_\lambda(q)=0$ si $q\epsilon\mathfrak{P}^0$ ou $\epsilon\mathfrak{P}^1$, $f_\lambda(q'q'')=\varphi(\bar\lambda(q'), f_\lambda(q''))+f(\bar\lambda(q'), \bar\lambda(q''))+\psi(f_\lambda(q'), \bar\lambda(q''))$.

Ces résultats sont explicités dans des cas particuliers: algèbres commutatives, algèbres associatives, algèbres de Lie, algèbres de Jordan. *H. Cartan* (Paris).

Lee, H. C. On Clifford algebras and their representations. I. Abstract properties of Clifford algebras. Ann. of Math. (2) 49, 760–773 (1948).

A Clifford algebra C_n over an algebraically closed field F of characteristic not two is the associative algebra of all polynomials in $u_1, \cdots, u_n$, where $u_i^2=1$, $u_ju_i=-u_iu_j$ for $i\neq j$. Then either $C_1=e_1F\oplus e_2F$ is the direct sum of two fields isomorphic to F, where $e_1=\frac{1}{2}(1-u_1)$, $e_2=\frac{1}{2}(1+u_1)$, or $u_1=1$ and the algebra degenerates. The algebra $C_2=F+u_1F+u_2F+u_1u_2F$ is a quaternion algebra isomorphic to the algebra of all two-rowed square matrices over F. If $n>2$ then C_n is immediately expressible as the direct product $C_2\times D_{n-2}$, where D_{n-2} is the Clifford algebra generated by $v_1, \cdots, v_{n-2}$ for $v_i=\sqrt{-1}\, u_1u_2u_{i-2}$. An immediate induction implies that if $n=2m$ then $C_n=C_2^{(1)}\times\cdots\times C_2^{(m)}$ is the direct product of m total matric algebras of degree two, and so is a total matric algebra of degree 2^m. The only irreducible representation of C_n is then C_n itself, and this yields all representations of C_n. If $n=2m+1$ then $C_n=C_{2m}^{(1)}\times C_2^{(1)}$ and $C_n=C_{2m}^{(1)}$ in case $C_1^{(2)}$ is degenerate. In the contrary case C_n is a direct sum $L\oplus M$ of two total matric algebras L and M of degree 2^m; every representation of C_n is a direct sum of a representation of L and one of M. We have then given a complete exposition, with proofs, of the theory of Clifford algebras, and the extension to the case where F is an arbitrary field of characteristic not two is immediate. The paper under review consists of a much more involved rederivation of the well-known results stated and proved above.

A. A. Albert (Chicago, Ill.).

THEORY OF GROUPS

Amato, V. Una proprietà caratteristica del gruppo del quadrato. Matematiche, Catania 1, 81–82 (1946).

The symmetric group of order $n!$ contains an element whose centralizer is the dihedral group of order $2n$ if and only if $n=4$.

C. C. MacDuffee (Madison, Wis.).

Piccard, Sophie. Deux propositions de la théorie des groupes de substitutions. Ann. Soc. Polon. Math. 21, 135–146 (1948).

In the first proposition it is assumed that $2\leqq m<n$ and the conditions to be satisfied by a substitution S on n symbols are determined such that $\{\mathfrak{S}_m, S\}=\mathfrak{S}_n$. In the second, $3\leqq m<n$, and similar conditions on S imply that $\{\mathfrak{A}_m, S\}=\mathfrak{A}_n$ or $\mathfrak{S}_n$ according as S is even or odd.

G. de B. Robinson (Toronto, Ont.).

Piccard, Sophie. Un théorème concernant le nombre des bases d'un sous-groupe transitif et primitif, à base du second ordre, du groupe symétrique. C. R. Acad. Sci. Paris 227, 745–747 (1948).

A basis of a noncyclic subgroup G of S_n consists of two substitutions P, Q which generate G. If G is transitive and primitive on the n symbols, the total number of bases of G is a multiple of m or $m/2$, where m is the number of substitutions of S_n which commute with G.

G. de B. Robinson (Toronto, Ont.).

Piccard, Sophie. Relations caractéristiques des bases du groupe symétrique. Mathematica, Timişoara 23, 88–100 (1948).

The author lists here the bases, consisting of two substitutions S, T which generate $\mathfrak{S}_n$ and the independent generating relations which hold for all possible such bases for

$n=3, 4, 5$, and the bases alone for $n=6$. The second half of the paper is devoted to a new proof of Hölder's theorem that S_6 is the only S_n which admits of outer isomorphisms.

G. de B. Robinson (Toronto, Ont.).

Bays, S. Sur la transitivité et la primitivité des groupes de substitutions. Comment. Math. Helv. **22** (1949), 17–30 (1948).

The transitivity or primitivity of a permutation group has heretofore been defined with regard to the individual symbols in terms of which the group is expressed. The author generalizes this definition to apply to pairs of symbols. If G is represented on the symbols $a, b, \cdots, l$, then these can always be divided into sets of imprimitivity for pairs in the following three ways: (1) (ab, ba), (cd, dc), $\cdots$; (2) $(ab, ac, \cdots, al)$, $(ba, bc, \cdots, bl)$, $\cdots$; (3) $(ba, ca, \cdots, la)$, $(ab, cb, \cdots, lb)$, $\cdots$. Other such divisions can only be made if G is doubly or triply transitive in the ordinary sense. The cyclic, metacyclic, alternating and symmetric groups are used to illustrate the theory. *G. de B. Robinson.*

Szép, J. Über die als Produkt zweier Untergruppen darstellbaren endlichen Gruppen. Comment. Math. Helv. **22** (1949), 31–33 (1948). $\mathfrak{K}$

Let $\mathfrak{G}$ be a finite group and let $\mathfrak{H}$ and $\mathfrak{K}$ be two permutable subgroups of $\mathfrak{G}$ such that $\mathfrak{H}\cup\mathfrak{K}=\mathfrak{G}$, $\mathfrak{H}\cap\mathfrak{K}=1$. Then for each $H\epsilon\mathfrak{H}$, $K\epsilon\mathfrak{K}$ there exists a unique $H'\epsilon\mathfrak{H}$ such that $H'^{-1}K\,H\epsilon\mathfrak{K}$, and the mapping $\pi_H:(K\rightarrow H'^{-1}KH)$ is a permutation of the elements of $\mathfrak{K}$. The correspondence $H\rightarrow\pi_H$ is a homomorphism of $\mathfrak{H}$ onto the group Π of all permutations π_H, whose kernel $\mathfrak{N}$ is the largest normal subgroup of $\mathfrak{G}$ contained in $\mathfrak{H}$. By remarking that $\mathfrak{N}\neq1$ if and only if the order of $\mathfrak{H}$ is greater than the order of Π, the author obtains the following criterion for the nonsimplicity of a group $\mathfrak{G}$ of the above type. If $\mathfrak{H}$ contains an element of order $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, p_i distinct primes, and if $p_1^{\alpha_1}+\cdots+p_r^{\alpha_r}$ is not less than the order of $\mathfrak{K}$, then $\mathfrak{G}$ is not simple. In particular, $\mathfrak{G}$ is not simple if $\mathfrak{H}$ contains an element of prime power order not less than the order of $\mathfrak{K}$.

S. A. Jennings (Vancouver, B. C.).

Frame, J. Sutherland. Group decomposition by double coset matrices. Bull. Amer. Math. Soc. **54**, 740–755 (1948).

The commuting matrices $V=(V^{is})$ of an intransitive permutation representation R of a given group G were previously studied by the author [same Bull. **49**, 81–92 (1943); these Rev. **4**, 192] with a view to obtaining information concerning the characters of G. If $R=\sum R^i$, where R^i is a transitive component of R, then $R^iV^{is}=V^{is}R^s$ and $V^{is}=\sum_\alpha c_\alpha^{is}V_\alpha^{is}$; the c_α^{is} are arbitrary parameters and V_α^{is} is a matrix consisting of zeros and 1's defined in terms of the double cosets obtained from the pair of subgroups H^i, H^s associated with the transitive representations R^i, R^s of G. If now R be completely reduced by transformation by a matrix U, this same transformation when applied to V yields $(U^i)^{-1}V_\alpha^{is}U^s=\Phi_\alpha^{is}$, whose coefficients are called double coset coefficients. These coefficients are expressible in terms of the coefficients of the irreducible representations F_j of G, and relations hold between them which yield the orthogonality relations between the coefficients of the F_j's as special cases. Similarly the double coset traces satisfy orthogonality relations which include the orthogonality relations between the characters of G. Moreover, these traces are expressible in terms of the characters of G and conversely.

Defining the double coset structure constants by the relation $H_\alpha^{rs}H_{\beta'}^{st}/h^s=\sum_\gamma c_{\alpha\beta\gamma}^{rst}H_\gamma^{rt}$, the author gave a theorem in his 1943 paper concerning the factoring of the determinant $|\sum_\alpha c_{\rho\sigma\alpha}^{tst}n_\alpha^{st}p_\alpha^t|$, where n_α^{st} is the index of the cross cut of H^t and $\gamma_\alpha^{-1}H^s\gamma_\alpha$ in G, and γ_α is any element of G. The quantities p_α^t are arbitrary parameters and ρ, σ represent row and column indices. This theorem is here corrected to read:

$$|\sum_\alpha c_{\rho\sigma\alpha}^{tst}n_\alpha^{st}p_\alpha^t|=n^sn^tA^{st}\prod_j f_j^{\mu_j^s\mu_j^t}|\sum_\alpha p_\alpha^t\Phi_{\alpha j}^{tt}|^{\mu_j^s},$$

where f_j is the degree of F_j and A^{st} and the μ's are certain integers. The application of this theorem to the calculation of group characters is based on the possible determination of the structure constants $c_{\rho\sigma\alpha}^{tst}$ when H^s and H^t are large so that R^s and R^t have relatively few irreducible components F_j in common. The author reviews the illustrative example of the former paper in the light of the revised theorem and considers also the group of the 27 lines on the general cubic surface, giving a method for obtaining the characters of the irreducible representations of degrees 1, 6, 20.

G. de B. Robinson (Toronto, Ont.).

Châtelet, Albert. Algèbre des relations de congruence. Ann. Sci. École Norm. Sup. (3) **64** (1947), 339–368 (1948).

The first two sections deal with certain relations including equivalence relations and follow quite closely the exposition in a previous paper by the author [Revue Sci. **85**, 579–596 (1947); these Rev. **9**, 407]. In the third section these ideas are illustrated on hypergroups which are defined as associative systems with many-valued multiplication and usually a unit element. To each subset A is associated the relations $xRy\rightleftarrows y\epsilon xA$, $xR'y\rightleftarrows y\epsilon Ax$. The author studies how the properties of these relations are reflected in the properties of A; for instance, in order that R be an equivalence relation A must be a reversible subhypergroup in the sense of Dresher and Ore [Amer. J. Math. **60**, 705–733 (1938)]. Invariant and permutable subhypergroups are introduced and the possibility of a Jordan-Hölder theorem is pointed out.

O. Ore (New Haven, Conn.).

NUMBER THEORY

Storchi, Edoardo. Sulle somme dei prodotti K a K dei primi n numeri della serie naturale. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 31–43 (1948).

Certain properties, mostly known, of numbers S_k^n, the sums of all combinations without repetition and k at a time of the first n natural numbers (and more familiar, apart from sign and ordering, as the Stirling numbers of the first kind) are developed. The following result seems of chief interest: for $k\leq[n/3]$, S_k^n is divisible by all primes lying between $n+1-k$ and $n+1$. *J. Riordan.*

Armellini, G. Osservazioni sui numeri perfetti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 9–14 (1948).

This paper contains proofs of various results concerning perfect numbers, such as those obtained by Euler and

Sylvester. The author simplifies Sylvester's proof that no odd perfect number with three prime factors can exist. He also proves that there exist no perfect numbers of the form $3k-1$, a result which appears to be due to Pepin [see L. E. Dickson, History of the Theory of Numbers, v. 1, Carnegie Institution of Washington, 1919, p. 28]. *R. A. Rankin.*

Richmond, Herbert W. A note upon an arithmetical property of quartic surfaces. J. London Math. Soc. **23**, 6–8 (1948).

Consider the equation (1) $ax^4+by^4=cz^4+dw^4$, where a, b, c, d are integers and $abcd$ is a square. The quartic tetrahedral surface represented by (1) (in homogeneous coordinates (x, y, z, w)) is such that the inflexional tangents in any of its rational points can be obtained rationally, and so from any known solution of (1) in rational numbers two other solutions may be deduced [cf. H. W. Richmond, same J. **19**, 193–194 (1944); B. Segre, same J. **19**, 195–200 (1944); Proc. Cambridge Philos. Soc. **41**, 187–209 (1945), § II; Proc. London Math. Soc. (2) **49**, 353–395 (1947), § VIII; these Rev. **7**, 244, 169; **9**, 135]. The present paper determines all the biaxial quartic surfaces (2) $F(x, y)=G(z, w)$, where $F(x, y)$ and $G(z, w)$ are two binary quartics with rational coefficients, to which the results above can be extended. There are only two types of such surfaces (2). For the first type $F(x, y)$ and $G(z, w)$ are the squares of two quadratics, so that the quartic (2) splits up into two quadrics. For the second type the cubic invariants of $F(x, y)$ and $G(z, w)$ vanish and a certain arithmetical condition is satisfied by the coefficients of these two forms; the quartic (2) is then a projective transform of (1) in the complex field, and is therefore included among those investigated arithmetically in the first of Segre's papers quoted above.

B. Segre (Bologna).

Mirsky, L. On a problem in the theory of numbers. Simon Stevin **26**, 25–27 (1948).

The following theorem is stated: If $r, s, k_1, \cdots, k_s, q_1, \cdots, q_s$ are given positive integers and $N(x)$ is the number of positive integers n not exceeding x such that all the integers $q_1n+k_1, \cdots, q_sn+k_s$ are rth-power-free, then

$$N(x)=Ax+O(x^{2/(r+1)+\epsilon}),$$

where A is a constant depending on the given parameters (which may be given by means of an infinite series or an infinite product). The author has given a proof earlier [Quart. J. Math., Oxford Ser. **18**, 178–182 (1947); these Rev. **9**, 80] of the special case in which $q_1=\cdots=q_s=1$ and remarks in the present paper that the extension of this proof to the general case is quite straightforward. However, the main purpose of this paper is to give a simple direct proof of the fact that lim inf $\{N(x)/x\}>0$ (and thus $A>0$) unless $N(x)$ is zero for all x. *P. T. Bateman* (Princeton, N. J.).

Lehmer, D. H. On the partition of numbers into squares. Amer. Math. Monthly **55**, 476–481 (1948).

Let $P_k(n)$ denote the number of partitions of n into k squares, i.e., the number of sets of integers $(x_1, \cdots, x_k)$ such that $0\leqq x_1\leqq\cdots\leqq x_k$ and $x_1^2+\cdots+x_k^2=n$. For every positive integer k except 3, the author determines all positive integers n for which $P_k(n)=1$. In the case $k=3$, since $P_3(4n)=P_3(n)$, it is sufficient to consider numbers n not divisible by 4. The author determines all such numbers n for which $P_3(n)=1$ up to 427, and shows that there are no more up to 10 000, and that there cannot be infinitely many. His list is therefore probably complete. *T. Estermann.*

Atkinson, F. V. A mean value property of the Riemann zeta-function. J. London Math. Soc. **23**, 128–135 (1948).

Let $\chi(s)=2^{1-s}\pi^{-s}\Gamma(s)\cos\frac{1}{2}\pi s$, the real function $\theta(t)$ be defined by $e^{2i\theta(t)}=\chi(\frac{1}{2}+it)$, and the real function $f(t)$ be defined by $f(t)=e^{i\theta(t)}\zeta(\frac{1}{2}+it)$. The author proves the following result. Let t' be defined as a function of t by the relation $\theta(t')-\theta(t)=\alpha$, α fixed and positive. Then

$$\int_0^T f(t)f(t')dt\sim(T\log T)\alpha^{-1}\sin\alpha$$

as $T\rightarrow\infty$. The proof uses the approximate functional equation for $\zeta(s)$. Then it is shown that the number of zeros of $\zeta(\frac{1}{2}+it)$, $T/2\leqq t\leqq T$, is greater than $AT/\log T$. The theorem above is analogous to a result obtained by Titchmarsh for a corresponding sum [Quart. J. Math., Oxford Ser. **5**, 98–105 (1934)]. *R. Bellman.*

Morrow, D. C. Universal quaternary quadratic forms. Bull. Amer. Math. Soc. **54**, 903–904 (1948).

A partial determination has been made of all positive quaternary quadratic forms $\sum c_{ij}x_ix_j$ (with integral coefficients c_{ii} and $2c_{ij}$) which represent all positive integers. Tables of the results so far obtained are being sent to the library of the American Mathematical Society.

G. Pall (Chicago, Ill.).

Swift, J. Dean. Note on discriminants of binary quadratic forms with a single class in each genus. Bull. Amer. Math. Soc. **54**, 560–561 (1948).

The author has verified that, beyond the largest known ($d=-3315$ if odd, $d=-4\cdot 1848$ if even), there are no negative discriminants with only one class of binary quadratic forms in each genus, at least up to $d=-10^7$. D. H. Lehmer's linear congruence machine was used. *G. Pall.*

Maass, H. Quadratische Formen über quadratischen Körpern. Math. Z. **51**, 233–254 (1948).

Let K be the real quadratic field $R(\sqrt{a})$, where a is a positive squarefree rational integer, and let I be the ring of integers of K. A quadratic form $\nu=\sum\alpha_{ik}\xi_i\xi_k$ $(\alpha_{ik}=\alpha_{ki}\subset I)$ is said to be an even positive form over K if, for all nontrivial sets $\xi\subset I$, $\nu\gg 0$ and $\nu\equiv 0 \pmod 2$. The author proves that there exists an even positive ν over K, whose determinant is a unit $\epsilon\subset I$, if and only if

$$\epsilon\gg 0,\quad n\equiv 0 \pmod 2,\quad (-1)^{n/2}\epsilon\equiv\alpha^2 \pmod 4,\quad \alpha\subset I.$$

These conditions lead to a method of constructing such forms, following Witt [Abh. Math. Sem. Hansischen Univ. **14**, 289–322, 323–337 (1941); these Rev. **3**, 100, 163], except for the case $a\equiv 1 \pmod 8$, which differs essentially from the others. The author carries out the constructions, studies vector diagrams whose groups of reflections lead easily to the groups of automorphisms of the associated forms, and obtains criteria for the equivalence of even positive ν over K. He has previously studied forms ν over $R(\sqrt{5})$ [Math. Ann. **118**, 65–84 (1941); these Rev. **3**, 272]. *R. Hull.*

Pall, Gordon. The minimum of a real, indefinite, binary quadratic form. Math. Mag. **21**, 255 (1948).

A simple evaluation is given of the first two Markoff minima of such forms. *H. S. A. Potter* (Aberdeen).

Macbeath, A. M. Non-homogeneous linear forms. J. London Math. Soc. **23**, 141–147 (1948).

Two more (geometrical) proofs are given of Minkowski's theorem that, if $a, b, \cdots, f$ are real with $ad-bc=\pm 1$, then

there are integers x, y such that

$$|(ax+by+e)(cx+dy+f)|\leqq\tfrac{1}{4}.$$

A new proof is also given of the following theorem, due to Chalk [Quart. J. Math., Oxford Ser. **18**, 215–227 (1947); these Rev. **9**, 413]: if L_i $(i=1, 2, \cdots, n)$ are n nonhomogeneous linear forms in n variables with real coefficients and unit determinant, then (i) integers (x) exist such that $L_i>0$ $(i=1, 2, \cdots, n)$ and $L_1L_2\cdots L_n\leqq 1$; (ii) integers (x) exist such that $L_1\geqq 0$, $L_i>0$ $(i=2, 3, \cdots, n)$ and $L_1L_2\cdots L_n<1$.
W. J. LeVeque (Cambridge, Mass.).

Hlawka, Edmund. Über Folgen von Quadratwurzeln komplexer Zahlen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 255–262 (1948).

This paper gives an extension to complex numbers of previous investigations of Mahler [Nieuw Arch. Wiskunde (2) **20**, 176–178 (1940); these Rev. **1**, 202] and Koksma [Nieuw Arch. Wiskunde (2) **20**, 179–183 (1940); these Rev. **1**, 202] on the approximability of real numbers by square roots of rational integers. Let $k(i)$ be the field of rational complex numbers, and let ξ be a complex number which is neither in $k(i)$ nor an integer in any quadratic field over $k(i)$. Then it is shown that for each $\epsilon>0$ there are infinitely many pairs x, $z\epsilon k(i)$ such that $|\xi+ez^{\frac12}+x|<(1+\epsilon)/4|z|$, where $e=\pm 1$ or $\pm i$ is a unit depending only on ξ, and where $\Re z^{\frac12}\geqq 0$, $\Im z^{\frac12}\geqq 0$. It is not shown that 4 is the best possible constant. A similar theorem is proved when z is restricted to be a complex integer, where now ξ is any number not in $k(i)$. When ξ is in $k(i)$, there is a constant $C(\xi)>0$ such that for each $z\neq 0$ with $\Re z^{\frac12}\geqq 0$, $\Im z^{\frac12}\geqq 0$ and for each $x\epsilon k(i)$, either $|\xi-z^{\frac12}+x|>C/|z|^{\frac12}$ or $\xi-z^{\frac12}+x=0$. *W. J. LeVeque.*

Hlawka, Edmund. Eine asymptotische Formel für Potenzsummen komplexer Linearformen. Monatsh. Math. **52**, 248–254 (1948).

Let $L_1(x, y)$ and $L_2(x, y)$ be linear forms with complex coefficients, whose determinant has absolute value unity. A theorem of Perron [Math. Ann. **103**, 533–544 (1930)] states that there exist integers x, y, not both zero, in the field $k(i)$ which satisfy $|L_1L_2|\leqq 3^{-\frac12}$, this being the best possible constant. The author deduces from Perron's proof that the integers can be so chosen that $|L_1|$ and $|L_2|$ are also absolutely bounded. Now let $M(\alpha)$ denote the lower bound of the numbers λ for which the inequality $|L_1|^\alpha+|L_2|^\alpha<\lambda$ is always soluble in integers x, y, not both zero, of $k(i)$. The author deduces estimates for $M(\alpha)$ for $0<\alpha<1$, analogous to those given by Mahler [Proc. Cambridge Philos. Soc. **40**, 107–116, 116–120 (1944); J. London Math. Soc. **18**, 233–238 (1943); these Rev. **6**, 119] for real linear forms. *H. Davenport* (London).

Cassels, J. W. S. The lattice properties of asymmetric hyperbolic regions. II. On a theorem of Davenport. Proc. Cambridge Philos. Soc. **44**, 145–154 (1948).

[For part I see the same vol., 1–7 (1948); these Rev. **9**, 335.] Let $R\geqq 0$, $S\geqq 0$. Put $\bar\omega(z)=16Sz+4p^2z^2$ for $z>0$, where $p=[(R+S)/z]$, $\bar\omega(0)=\bar\omega(0+0)=4(R+S)^2$. The only discontinuities of $\bar\omega$ are at the points $z=(R+S)/q$, $q=1, 2, \cdots$, where $\bar\omega(z)=\bar\omega(z-0)>\bar\omega(z+0)$. Let $f(x, y)$ be an indefinite quadratic form with the determinant Δ^2 $(\Delta>0)$. Let m, n be two coprime integers and put $f(m, n)=\Delta z$, supposing $f(m, n)\geqq 0$. Theorem: let $z\leqq 4R$, $\bar\omega(z)\geqq 1$; then to every point (x_0, y_0) there is a point $(x, y)\equiv(x_0, y_0)$ (mod 1) such that (1) $-\Delta S\leqq f(x, y)\leqq\Delta R$. If $\bar\omega(z)>1$, the equality signs in (1) are unnecessary except, perhaps, when either $z=4R$, $(x_0, y_0)\equiv(\frac12 m, \frac12 n)$ or $\bar\omega(z+0)\leqq 1$ or $RS=z=0$ [see also Davenport, Nederl. Akad. Wetensch., Proc. **49**, 815–821 (1946); **50**, 378–389, 484–491 (1947) = Indagationes Math. **8**, 518–524 (1946); **9**, 236–247, 290–297 (1947); these Rev. **8**, 444, 565; **9**, 79]. Applications. (I) The Euclidean algorithm is valid in the field $R(\surd t)$ for $t=2, 3, 6, 7, 11, 5, 13, 17, 21, 29, 33, 37, 41$. (II) Let θ be irrational, let α be any number which cannot be represented in the form $m'-n'\theta$ (m', n' integers). Suppose that there is a constant z and an infinite sequence of pairs of coprime integers (m_k, n_k), $|n_k|\to\infty$, such that either $\varphi_k=n_k(m_k-n_k\theta)\to z-0$, $0<z\leqq 4R$, $\bar\omega(z)>1$, or $\varphi_k\to z+0$, $0<z<4R$, $\bar\omega(z+0)\geqq 1$, or $\varphi_k\to+0$, $\bar\omega(0)>1$, or $\varphi_k\to+0$, $\bar\omega(0)=1$, $S\geqq R$. Then the inequalities $-S<n(m-n\theta-\alpha)<R$, $n\neq 0$, have an infinite number of solutions in integral m, n. [See also A. Khintchine, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 281–294 (1946); these Rev. **8**, 444.] [On p. 153, line 14, read $\bar\omega(z+0)$ instead of $\bar\omega(z)$.] *V. Jarník* (Prague).

Cassels, J. W. S. Lattice properties of asymmetric hyperbolic regions. III. A further result. Proc. Cambridge Philos. Soc. **44**, 457–462 (1948).

[Cf. the preceding review.] Let $f(x, y)=ax^2+bxy+cy^2$ be an indefinite quadratic form with $a>0$. (I) If x_0, y_0 are real numbers, there is a point $(x, y)\equiv(x_0, y_0)$ (mod 1) such that (1) $\frac14 e\leqq f(x, y)\leqq\frac14 a$; here $e=\min(0, e_1)$, $e_1=\min f(r, 1)$, where r runs through all integers [a classical result of Minkowski gives $|f(x, y)|\leqq\frac14(b^2-4ac)^{\frac12}$]. The second equality sign in (1) is unnecessary except perhaps if $(x_0, y_0)\equiv(\frac12, 0)$; the first one is unnecessary except perhaps if there is an integer r such that either $f(r, 1)=e_1<0$, $(x_0, y_0)\equiv(\frac12 r, \frac12)$ or $f(r, 1)=e_1=0$, $x_0\equiv ry_0$. (II) For special forms $f=k(x^2-xy)-y^2$ (k integer, $k>0$) (1) says that (2) $-\frac14\leqq f(x, y)\leqq\frac14 k$; but if k is odd, (2) can be replaced by the sharper relation (3) $-\frac14\leqq f(x, y)\leqq\frac14 k-\frac14(k+4)^{-1}$. Both (2) and (3) are best possible results. E.g.: if k is even, there is no point $(x, y)\equiv(\frac12, 0)$ for which $-\frac14(k+4)<f(x, y)<\frac14 k$ and no point $(x, y)\equiv(\frac12, \frac12)$ or $(0, \frac12)$ for which $-\frac14<f(x, y)<\frac14(2k-1)$. There is an analogous result concerning (3) (k odd). For other results concerning special forms f see Davenport [Nederl. Akad. Wetensch., Proc. **49**, 815–821 = Indagationes Math. **8**, 518–524 (1946); these Rev. **8**, 444]. [In (7) (p. 460) read $-\frac14$ instead of $\frac14$.] *V. Jarník* (Prague).

Niven, Ivan. Fermat's theorem for matrices. Duke Math. J. **15**, 823–826 (1948).

The L.C.M. of the orders of the elements of the group G_m of nonsingular matrices of order n with elements in $GF(p^m)$ is shown to be p^rq_n, where r is defined by $p^r\geqq n>p^{r-1}$, and $q_n=\text{L.C.M.}[p^m-1, p^{2m}-1, \cdots, p^{nm}-1]$. This completes and generalizes a theorem of J. B. Marshall [Proc. Edinburgh Math. Soc. (2) **6**, 85–91 (1939); these Rev. **1**, 199]. There is also given an algorithm for determining the orders of the individual elements of G_m. *N. H. McCoy.*

Carlitz, L. Representations of arithmetic functions in $GF[p^n, x]$. II. Duke Math. J. **15**, 795–801 (1948).

Let $GF[p^n, x]$ denote the ring of polynomials in the indeterminate x with coefficients in the Galois field $GF(p^n)$. In part I [same J. **14**, 1121–1137 (1947); these Rev. **9**, 337], the author proved that an arbitrary arithmetic function $f(A)$, $A\epsilon GF[p^n, x]$, $\deg A<r$, can be written in the form $f(A)=\sum^*\alpha_{GH}\epsilon_{GH}(-A)$, where $\alpha_{GH}=p^{-nr}\sum f(B)\epsilon_{GH}(B)$, the summation extends over all B with $\deg B<r$, and $\epsilon_{GH}(A)$ denotes a certain pth root of unity. The asterisk indicates

that the summation is restricted to a set which includes all ϵ_{GH} with $h=\deg H\leqq r/2$, while for $h>r/2$ certain ϵ_{GH} are superfluous. In various applications considerable difficulty is caused by the presence in the equation for $f(A)$ of ϵ's with $h>r/2$. The author shows in the present paper how this difficulty can in some problems be overcome by setting up a certain correspondence between ϵ's of degree $h>r/2$ and those with $h\leqq r/2$.

The rest of the paper is concerned with the evaluation of certain sums connected with the representation of a polynomial in $GF[p^n, x]$ as the sum of mth powers. It is shown that, if $r=mk$, $m\geqq 2$, $p\nmid m$, $h>(m-1)k$, then $\sum'\epsilon_{GH}(B^k)=0$, where the summation is over primary B of degree k. A formula for the sum over B of degree less than k is also derived. For $m=2$, these results have been proved in the author's earlier paper. *A. L. Whiteman.*

ANALYSIS

Verblunsky, S. On a problem of moments. Proc. Cambridge Philos. Soc. **45**, 1–4 (1949).

The author gives conditions which are necessary and sufficient for the existence of a nondecreasing solution $\sigma(t)$ of $c_i=\int t^i d\sigma(t)$, $i=0,\cdots,m$, where the integral extends either over $(-\infty,\infty)$ or over $(0,\infty)$. For example, for the first case, if $m=2n+1$ it is necessary and sufficient that there is a number c_{2n+2} such that the form $\sum_{p,q=0}^{n+1}c_{p+q}x_px_q$ is nonnegative; if $m=2n$ there must exist c_{2n+1}, c_{2n+2} satisfying the same requirement. The author remarks that his results can be deduced from known results on continued fractions, but his proofs are independent of the theory of continued fractions. *R. P. Boas, Jr.* (Providence, R. I.).

Teodorčik, K. On the theory of the synchronization of relaxational auto-oscillating systems. Učenye Zapiski Moskov. Gos. Univ. Fizika **95**, 3–8 (1946). (Russian)

The author reduces the physical problem to the mathematical problem of the convergence of the nth iterates of a certain system of functions derived by means of physical considerations. The convergence is then discussed in detail for a particular circuit. *R. Bellman.*

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, W. Sur un problème de M. N. Lusin. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 519–520 (1948).

The author gives an affirmative answer, under the hypothesis that $2^{\aleph_0}=\aleph_1$, to a question of Luzin, stated in Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 403–410 (1947); these Rev. **9**, 82. The question treated is (1) in the review cited. *E. Hewitt* (Seattle, Wash.).

Kesava Menon, P. On a class of perfect sets. Bull. Amer. Math. Soc. **54**, 706–711 (1948).

With $a_n>0$, $\sum a_n<\infty$, $R_n=\sum_{k>n}a_k$, the set S of numbers $\sum a_n'$, where $a_n'=0$ or a_n, is considered. If $a_n\geqq R_n$ (all n), S is shown to be perfect, of measure $\lim 2^nR_n$, and either a finite number of intervals or a totally disconnected set. Kakeya's theorem (if $a_n\leqq R_n$ for all n then S is an interval) is quoted, and the two theorems applied to the set $S(t)$ of numbers $\sum(-1)^{n-1}t^{i_n}$, $\{i_n\}$ being a finite or infinite sequence of increasing integers. *H. D. Ursell* (Leeds).

Lauwerier, H. A. Normal regions. Simon Stevin **26**, 28–31 (1948). (Dutch)

Let N be a plane bounded open point set and let R be the smallest open rectangle with sides parallel to the axes containing N. Then N is called x-normal if each line $x=x_0$ in R contains at least one point of N and exactly two boundary points $\eta_1(x_0)<\eta_2(x_0)$ of N. Similarly N is y-normal if $y=y_0$ in R contains at least one point of N, and exactly two boundary points $\xi_1(y_0)<\xi_2(y_0)$ of R. If N is both x-normal and y-normal, it is called normal. It is proved that if N is normal, then N is a normal region; that is, each line $x=x_0$ or $y=y_0$ has just one open interval in common with N, the functions $\eta_1(x_0)$, $\eta_2(x_0)$, $\xi_1(y_0)$, $\xi_2(y_0)$ are continuous, and the boundary of N has B-measure zero, so that N is J-measurable. *A. W. Goodman.*

Fort, M. K. A specialization of Zorn's lemma. Duke Math. J. **15**, 763–765 (1948).

Because of the essentially countable nature of many measure-theoretic arguments, transfinite methods are usually somewhat difficult to apply in measure theory. The author formulates and proves a simple consequence of Zorn's lemma in a form suitable for measure-theoretic applications. He applies his result to give short proofs of the Hahn and Jordan decompositions of a countably additive set function, the Lebesgue decomposition of one set function with respect to another, and the Radon-Nikodym theorem. The author's proof of the Hahn decomposition is considerably simpler than hitherto published proofs; his proof of the Radon-Nikodym theorem, however, is essentially the same as that of Yosida [Proc. Imp. Acad. Tokyo **17**, 228–232 (1941); these Rev. **4**, 247]. *P. R. Halmos* (Chicago, Ill.).

***Keldych, L. Sur la structure des ensembles mesurables B.** Trav. Inst. Math. Stekloff **17**, 75 pp. (1945). (Russian. French summary)

This monograph comprises a detailed exposition of results obtained over the past few years concerning the classification of Borel sets in the space I of irrational numbers lying in (0, 1). A number of these results have been previously announced [Rec. Math. [Mat. Sbornik] **41**, 187–220 (1934); Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **1938**, 125–135, 221–248; C. R. (Doklady) Akad. Sci. URSS (N.S.) **26**, 523–525 (1940); **28**, 675–677 (1940); **31**, 651–653 (1941); these Rev. **2**, 256; **3**, 226]. The family of all Borel sets in I is divided into $\aleph_1$ disjoint classes by the procedure of de la Vallée Poussin [see Lusin, Leçons sur les Ensembles Analytiques, Gauthier-Villars, Paris, 1930, pp. 53–54], these classes being designated by the symbols K_α $(0\leqq\alpha<\Omega)$. The aim of the monograph is to obtain a detailed description of all the sets in all the classes K_α. An initial reduction is obtained by considering only elements of K_α, which are defined as sets in K_α obtainable as countable intersections of sets of lower class $(\alpha\geqq 1)$. Since every set in K_α is the union of a countable family of pair-wise disjoint elements of K_α, this restriction is justified [see Lusin, loc. cit., pp. 76–77]. An element of the class K_1 is simply a closed set and an element of the class K_2 is a G_δ. Alexandroff and Urysohn have shown [Math. Ann. **98**, 89–106 (1927)] that there exists a certain fixed G_δ such that

every element of K_2 is the union of a countable family of sets of classes K_0 and K_1 and this fixed G_δ. The author's purpose is to generalize this result to all classes K_α.

The work is divided into three chapters. Chapter I deals with an analogue of Lusin's sieve, called an A-system, whereby an arbitrary element of the class K_α can be defined. Chapter II makes use of A-systems to define what are called canonical elements, in each class K_α. Technical restrictions on the defining A-system of an element of the class K_α suffice to make that element canonical. Various facts are proved concerning canonical elements: for example, two canonical elements of the class K_α are homeomorphic; any set homeomorphic to a canonical element of the class K_α is again such an element. The principal result concerning canonical elements is that any element of the class K_α is the set-theoretic union of a canonical element and a countable family of sets in classes K_β $(\beta<\alpha)$. The considerations of chapters I and II being based on the axiom of choice, it is of considerable interest actually to produce canonical elements in various classes. In chapter III, numerical examples are given of canonical elements in the classes K_α $(\alpha\leqq\omega)$, and a procedure is described for constructing numerical examples in K_α provided that a sequence $\{\beta_n\}_{n=1}^{\infty}$ of ordinal numbers is given such that $\lim_{n\to\infty}\beta_n=\alpha$. *E. Hewitt.*

Burkill, J. C. Differential properties of Young-Stieltjes integrals. J. London Math. Soc. 23, 22–28 (1948).

L'auteur donne d'abord un lemme relatif à l'intégrale d'une fonction d'ensemble, en rapport avec des résultats de A. Roussel [Bull. Soc. Math. France **70**, 1–30 (1942); ces Rev. **6**, 148], et de L. Tonelli [Ann. Scuola Norm. Super. Pisa (2) **8**, 309–321 (1939); ces Rev. **1**, 303]. Il en déduit une nouvelle démonstration du théorème suivant, dû à L. C. Young [Acta Math. **67**, 251–282 (1936)]: soient f et φ continues dans (a, b), et λ, μ telles que $|f(\xi)-f(x)|\leqq\lambda(h)$, $|\varphi(\xi)-\varphi(x)|\leqq\mu(h)$ pour $|\xi-x|\leqq h$; si $\int_0\lambda(t)\mu(t)t^{-2}dt$ converge, alors l'intégrale de Riemann-Stieltjes $\int_a^b fd\varphi$ existe. Il démontre ensuite deux propriétés, sous les mêmes hypothèses. (1) Posant $F(x)=\int_a^x fd\varphi$, pour tout ϵ, il existe δ tel que $|h|<\delta$ entraîne $F(x+h)-F(x)=f(x)[\varphi(x+h)-\varphi(x)]+\epsilon_1 h$, $|\epsilon_1|<\epsilon$. (2) Si, pour tout n, $f_n(x)$ et $\varphi_n(x)$ sont continues dans (a, b) et vérifient $|f_n(x+h)-f_n(x)|\leqq\lambda(h)$, $|\varphi_n(x+h)-\varphi_n(x)|\leqq\mu(h)$, et si f_n et φ_n convergent vers f et φ, alors $\int_a^b f_n d\varphi_n$ converge vers $\int_a^b fd\varphi$.

R. de Possel (Alger).

Giuliano, Landolino. Alcune proprietà delle trasformazioni assolutamente continue. Ann. Scuola Norm. Super. Pisa (2) 14 (1945), 91–98 (1948).

Continuing his recent work [same Ann. (2) **12** (1943), 161–172 (1947); these Rev. **9**, 339] on properties of absolutely continuous plane transformations, the author proves that if $\Phi: x=x(u, v)$, $y=y(u, v)$, $(u, v)\epsilon R$, is a biunique, continuous and absolutely continuous plane transformation defined on a simply connected bounded Jordan region R in the (u, v)-plane, then a necessary and sufficient condition for the inverse transformation Φ^{-1} to be absolutely continuous is that the generalized Jacobian $H(u, v)$ relative to Φ be different from zero almost everywhere on R^0, the interior of R. Moreover, if $H'(x, y)$ is the generalized Jacobian relative to the absolutely continuous inverse transformation Φ^{-1}, then $H'[x(u, v), y(u, v)]H(u, v)=1$ almost everywhere on R^0. To prove the last assertion the author shows that if a biunique, continuous and absolutely continuous plane transformation is followed by a continuous and absolutely continuous plane transformation, then the composite mapping is absolutely continuous and the generalized Jacobian of the composite map is equal almost everywhere to the product of the generalized Jacobians of the factor maps. *R. G. Helsel* (Columbus, Ohio).

Theory of Functions of Complex Variables

Tolstov, G. P. On curvilinear integrals in the sense of Lebesgue. Mat. Sbornik N.S. 23(65), 53–76 (1948). (Russian)

In terms of "admissible sets in a region" the author characterizes functions satisfying Lipschitz conditions and generalizes Morera's theorem. Let $C: z=\varphi(s)=(x(s), y(s))$ be a curve in the complex plane with arc length as parameter. The line integral of $Pdx+Qdy$ is defined to be the Lebesgue integral, if it exists, $\int_0^L(P(\varphi(s))dx/ds+Q(\varphi(y))dy/ds)ds$, where L is the length of the curve. For such a C and for each subset E of the plane, m_CE is the Lebesgue measure of the set of s for which $\varphi(s)$ is in E. Then C is called an admissible path through E in G if G is a region, C is a simple rectifiable curve contained, except for its end points, in G, and $m_CE=0$. The set E is admissible in G if there exists a number K such that for each rectangle $R\subset G$, and for each pair A, B of points in or on the boundary of R, there exists an admissible path C through $R\cap E$ in R such that $m_CE\leqq K|A-B|$. Theorem 1. If the rectangle R contains an admissible set E and a function F has bounded variation along every admissible path through E in R, then F has at most a countable number of singularities in R and these are removable. Theorem 4. For a continuous F the following are equivalent conditions: (a) the hypothesis of theorem 1; (b) F of bounded variation in the sense that for every sequence of non-overlapping circular disks D_n such that $\sum_n(\text{diam } D_n)<\infty$, $\sum_n(\text{osc } F \text{ on } D_n)<\infty$; (c) F absolutely continuous in the corresponding sense; (d) F satisfies a Lipschitz condition. Theorem 5. If R contains the admissible set E and the line integral of $Pdx+Qdy$ along admissible contours depends only on the end points, it defines an $F(z)=\int_{z_0}^{z}Pdx+Qdy$ satisfying a Lipschitz condition. Theorem 7. If the planar measure of E is zero so that E is admissible in a region G, if for every admissible closed path C through E lying entirely in G, we have $\int_C f(z)dz=0$, and if f is measurable in G, then f differs from an analytic function on at most a set of plane measure zero. Theorems 1–6 generalize to n dimensions. *M. M. Day* (Princeton, N. J.).

Cooper, J. L. B. Functions analytic in a half-plane. J. London Math. Soc. 23, 84–92 (1948).

Representation theorems of the Poisson-Stieltjes type are established for functions analytic in a half-plane. Use is made of a modified form of the well-known theorem of F. Riesz concerning linear functionals on the space of functions continuous on a bounded closed interval and the Cauchy integral formula. The principal result may be stated as follows. Let $f=p+iq$ be analytic in the upper-half $w(=u+iv)$-plane and satisfy (a) $M(v)=\sup_{-\infty<u<\infty}|f(u+(v))|<\infty$ for $v>0$ and $\lim_{v\to\infty}M(v)=0$, (b) there exists α, $0\leqq\alpha\leqq 2$, such that

$$\int_{-\infty}^{\infty}(1+|u|)^{-\alpha}|p(u, v)|du\leqq M_v,$$

where M_v is bounded for $0<v\leqq v_0$. Then there exists a func-

tion ρ of bounded variation on any finite interval such that

$$\int_{-\infty}^{\infty}(1+|x|)^{-\alpha}|d\rho(x)|\leqq\limsup_{v\to 0} M_v,$$

and

$$\rho(u_2)-\rho(u_1)=\lim_{v\to 0}\int_{u_1}^{u_2}p(u,v)du$$

for any pair of real numbers u_1 and u_2. If $\alpha\leqq 1$, then

$$f(w)=(\pi i)^{-1}\int_{-\infty}^{\infty}(x-w)^{-1}d\rho(x), \qquad \mathfrak{J}(w)>0,$$

and if $\alpha\leqq 2$,

$$p(w)=\pi^{-1}\int_{-\infty}^{\infty}|x-w|^{-2}vd\rho(x), \qquad \mathfrak{J}(w)>0.$$

A converse theorem is also given. *M. Heins.*

Zygmund, A. On a theorem of Hadamard. Ann. Soc. Polon. Math. 21, 52–69 (1948).

The author generalizes Hadamard's theorem to the effect that the circle of convergence of the gap series (1) $\sum c_k z^{n_k}$ ($\liminf n_{k+1}/n_k>1$) is a natural boundary for the function represented by the series (1) [Hadamard, J. Math. Pures Appl. (4) 8, 101–186 (1892), cf. § 13]. The generalization concerns series of the form

$$\text{(2)}\qquad \tfrac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos n_kx+b_k\sin n_kx)$$

(a_k, b_k real) and the associated harmonic functions

$$\text{(3)}\qquad u(r,x)=\tfrac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos n_kx+b_k\sin n_kx)r^{n_k}.$$

The following are the basic results. If a series (2) has Hadamard gaps and converges on a set E of positive measure, and if there exists a function $\varphi(x)$ which is analytic on some interval containing the set E and which coincides with the sum of (2) on E, then the function (3) is harmonic in some circle $|r|<1+\epsilon$. If (2) has Hadamard gaps and converges to zero on a set of positive measure, its coefficients are zero. These statements remain true if the hypothesis of convergence is replaced by that of summability by a linear method; and if the hypothesis of Hadamard gaps is replaced by the hypothesis that (2) is the sum of finitely many series with Hadamard gaps. *G. Piranian* (Ann Arbor, Mich.).

Hayman, W. K. Some inequalities in the theory of functions. Proc. Cambridge Philos. Soc. 44, 159–178 (1948).

The paper concerns the rate of growth of the maximum modulus $M(\rho,f)$ of a function f which in $|z|<1$ does not take a sequence of values $w_n\to\infty$. The results seem very accurate, but their formulation is extremely complicated. We can only quote the following consequence. The fact that there are arbitrarily large exceptional values implies that $\limsup_{\rho\to 1}(1-\rho)\log M(\rho,f)=0$ and $\liminf_{\rho\to 1}(1-\rho)^{\frac{1}{2}}\log M(\rho,f)=0$, and these are the best possible results of this nature. The proofs are essentially based on subordination and skillful handling of technical points. *L. Ahlfors* (Cambridge, Mass.).

Komatu, Yûsaku. On conformal slit-mapping of a circular ring. Math. Japonicae 1, 24–27 (1948).

In a previous note [Proc. Phys.-Math. Soc. Japan (3) 25, 1–42 (1943); these Rev. 7, 514] the author derived for doubly-connected slit domains a differential equation corresponding to that of Löwner for a simply-connected slit domain. In the equation for doubly-connected slit domains there appears a function $\gamma(q)$, $|\gamma(q)|=1$, which is the analogue of Löwner's function $\kappa(t)$, $|\kappa(t)|=1$. In his note [Proc. Imp. Acad. Tokyo 17, 11–17 (1941); these Rev. 2, 276] the author proved that in the case of analytic slits the derivative of $\arg\kappa(t)$ with respect to t for $t<t_0$ is equal to $3\rho(t)$, where $\rho(t)$ is the curvature of the slit in the plane of Löwner's function $f(z,t)$ at the point where this slit issues into the interior of the unit circle. In the present note the author extends this result to doubly-connected slit domains, where he finds that

$$d\arg\gamma(q)/d\log q=-3\rho(q)+4\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\sin n(\arg\gamma(q)).$$

H. L. Royden (Stanford University, Calif.).

Goluzin, G. M. On the coefficients of univalent functions. Mat. Sbornik N.S. 22(64), 373–380 (1948). (Russian)

Let $f(z)=z+c_2z^2+c_3z^3+\cdots+c_nz^n+\cdots$ be regular and univalent in $|z|<1$. Littlewood [Proc. London Math. Soc. (2) 23, 481–519 (1924)] has shown that $|c_n|<en$. The author proves that $|c_n|<\frac{3}{4}en$. A slightly better bound is given by Bazilevič in the paper reviewed below.

Let $m(r)=\sum_{n=1}^{\infty}|c_n|^2r^n$. The author proves the inequality

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|f(re^{i\varphi})|^{\lambda}d\varphi\leqq\lambda\int_0^r m'(\rho^2)^{\lambda/2}\rho^{\lambda-1}d\rho.$$

By means of an application of Hölder's inequality and integration by parts, he shows that if $m(r)\leqq m_0(r)$, where $m_0(0)=0$ and $m_0'(r)$ is increasing, then in the integral on the right we may replace $m'(\rho^2)$ by $m_0'(\rho^2)$. Using the inequality $|(F(\zeta_1)-F(\zeta_2))/(\zeta_1-\zeta_2)|\geqq 1-\rho^{-2}$, $|\zeta_1|=|\zeta_2|=\rho>1$, where $F(\zeta)=1/f(1/\zeta)$ [Goluzin, Rec. Math. [Mat. Sbornik] N.S. 19(61), 183–202 (1946); 21(63), 83–117 (1947); these Rev. 8, 325; 9, 421], it is shown that $m(r)\leqq 1/\int_r^1(1-\rho)^2\rho^{-3}d\rho=m_0(r)$. Taking $\lambda=1$, the bound for c_n is obtained from the inequality

$$|c_n|\leqq(2\pi r^n)^{-1}\int_{-\pi}^{\pi}|f(re^{i\varphi})|d\varphi\leqq\int_0^r m_0'(\rho^2)^{\frac{1}{2}}d\rho.$$

D. C. Spencer (Stanford University, Calif.).

Bazilevič, I. E. Improvement of estimates for the coefficients of univalent functions. Mat. Sbornik N.S. 22(64), 381–390 (1948). (Russian)

Let $f(z)=z+c_2z^2+c_3z^3+\cdots+c_nz^n+\cdots$ be a function which is regular and univalent (schlicht) in $|z|<1$. It was proved by Littlewood [reference in the preceding review] that $|c_n|<en$ ($n=2,3,\cdots$). By refining Littlewood's method, Landau [Math. Z. 30, 635–638 (1929)] showed that $\limsup\alpha_n/n\leqq(\frac{1}{2}+\pi^{-1})e<0.82e$, where $\alpha_n=\sup|c_n|$. The author introduces Goluzin's inequality $|(F(\zeta_1)-F(\zeta_2))/(\zeta_1-\zeta_2)|\geqq 1-\rho^{-2}$ [see the preceding review] into the Littlewood-Landau method, and establishes the following estimates:

$$\limsup\alpha_n/n\leqq(e/\pi)\int_0^{\pi}x^{-1}\sin x\,dx<0.59e;$$

$$|c_n|<1.924n,\ n\geqq 4;\quad |c_n|<1.759n,\ n\geqq 10;$$

$$|c_n|<1.621n,\ n\geqq 100.$$

D. C. Spencer (Stanford University, Calif.).

Biernacki, M. Sur une inégalité entre les moyennes des dérivées logarithmiques. Mathematica, Timişoara 23, 54–59 (1948).

Let $f(z)$ be regular and single-valued on the circumference $|z|=r>0$, and suppose that neither f nor f' vanishes on

$|z|=1$. The author shows that

$$\int_0^{2\pi}|\Re(zf'(z)/f(z))|d\theta \leqq \int_0^{2\pi}|\Re(1+zf''(z)/f'(z))|d\theta.$$

The case of equality is discussed. Applying a theorem of Zygmund [Fund. Math. **13**, 284–303 (1929)], he obtains the inequality

$$\int_0^{2\pi}|zf'(z)/f(z)|d\theta \leqq \{1+A \max_{|z|=r}\log^+|\Re(zf'(z)/f(z))|\} \times \int_0^{2\pi}|\Re(1+zf''(z)/f'(z))|d\theta+B,$$

where A and B are constants. *D. C. Spencer.*

Rosenbloom, Paul Charles. L'itération des fonctions entières. C. R. Acad. Sci. Paris **227**, 382–383 (1948).

L'auteur étudie les points fixes des itérées des fonctions entières. Si $f(z)$ est une fonction entière, l'équation $f(z)=z$, qui donne les points fixes, peut ne pas avoir de solutions ou n'en avoir qu'un nombre fini. Utilisant les résultats de Nevanlinna et la forme précise du théorème de Schottky, l'auteur donne ce théorème général: si f et g sont des fonctions entières permutables, c'est-à-dire $f(g(z))=g(f(z))$, et si $f(z)$ et $f(g(z))$ n'ont qu'un nombre fini de points fixes, les deux fonctions f et g sont des polynomes. Il s'ensuit que, si l'une des itérées de f n'a qu'un nombre fini de points fixes, f est un polynome. Il faut noter que la première proposition de l'auteur: l'itérée $f(f(z))$ a toujours des points fixes sauf si $f(z)$ se réduit à $z+$constante, est contenue dans ce résultat de Fatou [Acta Math. **47**, 337–370 (1926), p. 346]: si $f(z)$ non polynomiale n'a qu'un nombre fini de points fixes, $f(f)$ en a une infinité. *G. Valiron* (Paris).

Marden, Morris. On the zeros of the derivative of an entire function of finite genre. Proc. Nat. Acad. Sci. U. S. A. **34**, 405–407 (1948).

L'auteur donne une généralisation du théorème de Gauss-Lucas pour les fonctions entières $E(z)$ de genre fini p. Parmi les applications qu'il fait citons le resultat suivant. Soient α et β deux nombres non négatifs, tels que $\alpha\leqq\beta$, $\alpha+\beta=\pi/(p+1)$. Si les zéros de la fonction entière $E(z)$ de genre p se trouvent dans le secteur $|\arg z|\leqq\alpha$ la dérivée $E'(z)$ a au plus p zéros dans le secteur $|\arg(-z)|\leqq\beta$.
N. Obrechkoff (Sofia).

Boas, R. P., Jr. Basic sets of polynomials. I. Duke Math. J. **15**, 717–724 (1948).

A set of polynomials $\{p_n(z)\}$ is basic if every polynomial is a finite linear combination of the $p_n(z)$. If every function regular in $|z|<R$ (or in $|z|\leqq R$) has a p_n-expansion that is uniformly convergent in $|z|\leqq R'$ for every $R'<R$ (or in $|z|\leqq R$), then $\{p_n(z)\}$ is called effective in $|z|<R$ (or in $|z|\leqq R$). Again, if every entire function of a given class C has a p_n-expansion, uniformly convergent in every circle, then $\{p_n(z)\}$ is effective for class C. Various results on effectivity of basic sets are found here, among them the following. (I) Let $\lambda_n\neq0$ be a sequence of numbers with $\lambda_n\sim(\sigma e\rho/n)^{-n/\rho}$ (for fixed σ and ρ). Let $p_n(z)=\sum p_{nk}z^k$ be a basic set of polynomials, and define $q_n(z)=\sum p_{nk}(\lambda_k/\lambda_n)z^k$. If $\{q_n(z)\}$ is effective in $|z|\leqq1$ then $\{p_n(z)\}$ is effective for the class of entire functions of growth less than order ρ, type σ. (II) Let $H_2(R)$ be the class of functions $f(z)$ regular in $|z|<R$ and for which $\int_0^{2\pi}|f(re^{i\theta})|^2d\theta$ is bounded for $r<R$. If $|p_{nk}|\leqq\beta_{n-k}$ $(n, k=0, 1, \cdots)$, where $\sum_1^\infty\beta_kR^{-k}<1$, then every function $f(z)$ of $H_2(R)$ has a unique representation $f(z)=\sum_0^\infty c_np_n(z)$, which (a) converges in mean square on $|z|=R$, and (b) converges uniformly and absolutely in $|z|\leqq r$ for all $r<R$. As corollaries there appear numerous earlier results of others on effectivity, for which references are given; and there is a result on Laguerre polynomial expansions. *I. M. Sheffer* (State College, Pa.).

Fourier Series and Generalizations, Integral Transforms

Tveritin, A. N. An application of the theory of moments to the theory of trigonometric series. Doklady Akad. Nauk SSSR (N.S.) **61**, 985–988 (1948). (Russian)

Let $a_1, a_2, \cdots$ be an absolutely monotone sequence of numbers, a condition equivalent to the equations $a_n=\int_0^1\rho^{n-1}d\sigma(\rho)$ $(n=1, 2, \cdots)$, where $\sigma(\rho)$ is nondecreasing and bounded. The purpose of the note is to give a rigorous proof of the formally obvious equation

$$\sum_{n=1}^\infty a_n\sin nx=\int_0^1\frac{\sin x d\sigma(\rho)}{\rho^2-2\rho\cos x+1}.$$

The author says that the corresponding formula for $\sum_1^\infty a_n\cos nx$ is also valid but that the argument then becomes more complicated. *A. Zygmund* (Chicago, Ill.).

Timan, A. F. On the Lebesgue constants for certain methods of summability. Doklady Akad. Nauk SSSR (N.S.) **61**, 989–992 (1948). (Russian)

Let $\{\alpha_n\}$ be any sequence of real numbers not exceeding π in absolute value. Let $S_n(x, f)$ be the partial sums of the Fourier series of $f(x)$, and let $L_n(\alpha_n)$ be the Lebesgue constants corresponding to the method of summability defining $\lim_{n\to\infty}S_n(x, f)$ as $\frac12\lim_{n\to\infty}\{S_n(x, f)+S_n(x+\alpha_n, f)\}$[see Rogosinski, Math. Ann. **95**, 110–134 (1925); S. Bernstein, C. R. Acad. Sci. Paris **191**, 976–979 (1930)]. It is shown that, with an error $O(1)$, $L_n(\alpha_n)$ is equal to

$$4\pi^{-2}(1+n|\alpha_n|)\{|\alpha_n|+n^{-1}\}^{-|\cos\frac12(2n+1)\alpha_n|}.$$

A. Zygmund (Chicago, Ill.).

Sargent, W. L. C. On the summability (C) of allied series and the existence of (CP) $\int_0^\pi\frac{f(x+t)-f(x-t)}{t}dt$. Proc. London Math. Soc. (2) **50**, 330–348 (1948).

The well-known conditions for the Cesàro summability of Fourier-Lebesgue series and of their conjugates [see, e.g., L. S. Bosanquet and J. M. Hyslop, Math. Z. **42**, 489–512 (1937), and the references given there] are extended here to the case of functions integrable in the Cesàro-Perron ($C_\lambda P$) sense. We quote a few examples of the results obtained. Let $f(x)$ be of period 2π, and integrable $C_\lambda P$, where $\lambda\geqq0$. Let $\varphi(t)=\frac12\{f(x+t)+f(x-t)\}$, $\psi(t)=\frac12\{f(x+t)-f(x-t)\}$ and let $\sum A_n(x)$ and $\sum B_n(x)$ denote respectively the Fourier series of f and the conjugate series. (1) If $\alpha\geqq0$ and $(C, \alpha)\lim_{t\to0}\varphi(t)=s$, then $\sum A_n(x)$ is summable (C, β) to s, where $\beta>\alpha$ if $\alpha\geqq\lambda+1$, and $\beta\geqq\lambda+1$ if $0\leqq\alpha<\lambda+1$. (2) If $\alpha\geqq0$, $(C, \alpha)\lim_{t\to0}\psi(t)=0$, and

$$(C)\lim_{t\to0}\pi^{-1}\int_t^\pi\psi(u)\cot\tfrac12u\,du=s,$$

then $\sum B_n(x)$ is summable (C, β) to s, where β satisfies the

same conditions as in (1). (3) The integral

$$\int_0^\pi t^{-1}\{f(x+t)-f(x-t)\}dt$$

exists as a $C_\lambda P$ integral for almost every x.

A. Zygmund (Chicago, Ill.).

Lozinskiĭ, S. M. The spaces $\tilde{C}_\omega$ and $\tilde{C}_\omega{}^*$ and the convergence of interpolation processes in them. Doklady Akad. Nauk SSSR (N.S.) 59, 1389–1392 (1948). (Russian)

The paper is concerned with the space C of functions which are continuous and periodic with period 2π. Let $\omega(u)$ be a modulus of continuity, C_ω the class of f's such that $\sup_{x,u}|f(x+u)-f(x)|/\omega(u)$ is finite, $C_\omega{}^*$ the class of f's such that $\lim_{u\to 0+}\sup_x|f(x+u)-f(x)|/\omega(u)=0$. Given a sequence M of finite sets of points $x_k^{(n)}$, $k\leqq 2n+1$, $n=0, 1, \cdots$, let $l_k^{(n)}$ denote the trigonometric polynomial of degree n such that $l_k^{(n)}(x_k^{(n)})=1$ and $l_k^{(n)}(x_i^{(n)})=0$ for $i\neq k$. We write

$$\Lambda_n(M)=\sup_x\sum_{k=1}^{2n+1}|l_k^{(n)}(x)|,\quad U_n(M,f,x)=\sum_{k=1}^{2n+1}f(x_k^{(n)})l_k^{(n)}(x).$$

A theorem of Bernstein asserts that for any M, $\Lambda_n(M)>A\log n$ for all n and an absolute constant A. The main theorem is the following. If (i) $\limsup\omega(1/n)\log n>0$ or (ii) $\liminf_{u\to 0+}\omega(u\omega(u))/\omega(u)>0$, then a necessary and sufficient condition that $U_n(M,f,x)$ converge uniformly to f for every $f\epsilon C_\omega$ is (iii) $\lim_{n\to\infty}\Lambda_n(M)\omega(1/n)=0$. By the above theorem of Bernstein (i) and (iii) are obviously contradictory, so it seems to the reviewer that, unless a typographical error is involved, (i) should be omitted as an alternative condition. There are some further theorems and remarks in the same direction. The paper contains no proofs.

František Wolf (Berkeley, Calif.).

Lozinskiĭ, S. M. On the strong convergence of interpolation processes. III. Doklady Akad. Nauk SSSR (N.S.) 60, 961–964 (1948). (Russian)

[For parts I and II cf. C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 203–206 (1940); 30, 386–390 (1941); these Rev. 2, 195, 284.] With notations as in the preceding review, it is stated that, for any M,

$$(2n+1)^{-1}\sum_{k=1}^{2n+1}\operatorname*{Var}_{0\leqq x\leqq 2\pi}l_k^{(n)}(x)\geqq(4/\pi)\log n.$$

The main theorem is the following. If (i) $\mu(0)=0$, $\mu(u)\geqq 0$, convex and increasing for $u\geqq 0$, (ii) $\lim_{u\to 0+}\mu(u)/u=0$, $\lim_{u\to\infty}\mu(u)/u=\infty$ and $\liminf_{u\to\infty}\mu(u)/u\log u=0$, then to every M there exists a periodic, absolutely continuous function such that

$$\int_0^{2\pi}\mu[|f'(x)|]dx<\infty,\qquad \limsup_{n\to\infty}\int_0^{2\pi}|U_n'(M,f,x)|dx=\infty.$$

The last of conditions (iii) is necessary as has been proved before [Lozinskiĭ, Rec. Math. [Mat. Sbornik] N.S. 14(56), 175–268 (1944); these Rev. 6, 264]. An analogous set of results is given for interpolation polynomials in the interval $(-1, 1)$. The paper contains no proofs. *František Wolf.*

Lozinskiĭ, S. M. On a class of linear operations. Doklady Akad. Nauk SSSR (N.S.) 61, 193–196 (1948). (Russian)

In addition to the definitions given in the second preceding review we have to introduce still the following definitions: $U_n(f)$, $f\epsilon C$, is called a polynomial linear operation of order n if $U_n(f)$ is a trigonometric polynomial of order n and if, for f a trigonometric polynomial of order n, $U_n(f)=f$. Further let

$$s_n(f;x)=\pi^{-1}\int_0^{2\pi}f(x+t)\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}dt,$$

$$L_n=\pi^{-1}\int_0^{2\pi}\left|\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right|dt.$$

Then the following theorems are enunciated. (1) For any such U_n from C to C, the bound $|||U_n|||\geqq L_n$ and the lower bound is actually attained by $U_n=s_n$. (2) If $\limsup\omega(1/n)\log n=\infty$ and U_n is a sequence of polynomial operators, then there exists an $f\epsilon C_\omega{}^*$ such that $\limsup_{n\to\infty}\|U_n(f)\|=\infty$. If $\limsup\omega(1/n)\log n>0$, then there exists an $f\epsilon C_\omega$ such that $\limsup_n\|f-U_n(f)\|>0$. Theorem 3 asserts that theorem 1 remains valid if C is replaced by the space L of integrable periodic functions with the usual definition of norm. (4) If $\mu(u)$ is as in the preceding review and L^μ is the space of functions for which $\|f\|_\mu=\int_0^{2\pi}\mu[|f(x)|]dx$ is finite and U_n is a sequence of polynomial operations from L^μ to L, then there exists an $f\epsilon L^\mu$ such that $\limsup_n\|U_n(f)\|=\infty$. (5) If $\liminf\mu(2u)/\mu(u)=2$ and U_n is a sequence of polynomial operators from L^μ to L^μ, then there exists an $f\epsilon L^\mu$ such that $\limsup_n\|U_n(f)\|_\mu=\infty$. The remainder of the paper is devoted to stating the analogous theorems in which ordinary polynomials in $(-1, 1)$ replace trigonometric polynomials of period 2π. No proofs are given. *František Wolf.*

Berman, D. L. On an interpolation process. Doklady Akad. Nauk SSSR (N.S.) 61, 5–8 (1948). (Russian)

Let $\|x_m^{(n)}\|$, $m=1, \cdots, n$; $n=1, 2, \cdots$, be a matrix of numbers situated in the closed interval $-1\leqq x\leqq 1$ and consider the polynomials of interpolation defined by

$$N_n(f,x)=\sum_{k=1}^n l_k^{(n)}(x)\frac{2\sin(2h+1)\sin^{-1}\frac{1}{2}(x-x_k^{(n)})}{(2h+1)(x-x_k^{(n)})}f(x_k^{(n)}),$$

where $l_k^{(n)}(x)$ is the polynomial of degree n equal to unity if $x=x_k^{(n)}$ and to zero if $x=x_\nu^{(n)}$, $\nu\neq k$, and where h is a function of n taking only integral values and satisfying $0<\delta_1<2h/n<\delta_2<1$. The author proves the following theorem. Suppose that corresponding to any positive number ϵ there is a number c_1 such that

$$(*)\qquad \sum_{k=1}^n[l_k^{(n)}(x)]^2\leqq c_1,$$

$$-1+\epsilon\leqq x\leqq 1-\epsilon;\ n=1, 2, \cdots,$$

and suppose that $f(x)$ is continuous in the closed interval $[-1, 1]$. Then the interpolating polynomials $N_n(f, x)$, whose points of interpolation are those of the given matrix, converge to $f(x)$ at any point x_0 of the open interval $(-1, 1)$. The convergence is uniform in any closed interval $[-1+\epsilon, 1-\epsilon]$, $0<\epsilon<1$. The author remarks that condition (*) is satisfied, in particular, if the points of interpolation are the Jacobi abscissae with parameters $\alpha\geqq -1$ and $\beta\geqq -1$, for then Fejér has shown that

$$\sum_{k=1}^n[l_k^{(n)}(x)]^2<\max(-1/\alpha, -1/\beta).$$

A. C. Offord (London).

★Tricomi, Francesco. Serie Ortogonali di Funzioni. S.I.E. Istituto Editoriale Gheroni, Torino, 1948. 343+9 pp.

This course of lectures contains a brief introduction to orthogonal functions in general, an account of important topics in Fourier series, and a discussion (nearly half the

book) of the classical orthogonal polynomials, in which both properties of the individual polynomials and convergence theorems are given. For the topics included, the treatment is modern and quite thorough.

R. P. Boas, Jr. (Providence, R. I.).

Senčiščev, N. A. On the theory of orthogonal series. Doklady Akad. Nauk SSSR (N.S.) 62, 31–33 (1948). (Russian)

Let $\varphi_1(x), \varphi_2(x), \cdots$ be an orthonormal system in (a, b). A necessary and sufficient condition that from $\{\varphi_n\}$ we can select a subsequence $\varphi_{n_1}, \varphi_{n_2}, \cdots$ $(n_1<n_2<\cdots)$ such that the convergence of $\sum c_k\varphi_{n_k}$ on any set of positive measure implies $\sum c_k^2<\infty$, is that (*) $\limsup_{n\to\infty}\int_E|\varphi_n|dx>0$ for every set $E\subset(a, b)$ and of positive measure. If (*) is satisfied, then we can select $\{\varphi_{n_k}\}$ in such a way that already the boundedness of the partial sums of $\sum c_k\varphi_{n_k}$ at every point of a set of positive measure implies $\sum c_k^2<\infty$. The results are generalizations of theorems of Marcinkiewicz [Studia Math. 8, 1–27 (1939)].

A. Zygmund.

Boas, R. P., Jr., and Pollard, Harry. The multiplicative completion of sets of functions. Bull. Amer. Math. Soc. 54, 518–522 (1948).

The set $\{f_n(x)|-\infty\leqq a\leqq x\leqq\infty,\ f_n(x)\epsilon L_2\}$ is multiplicatively complete if, for some $m(x)$, $\{m(x)f_n(x)\}$ is complete. The authors give some results, but necessary and sufficient conditions are still lacking. Thus if $\{f_n(x)\}$ is orthonormal and has finite deficiency, it is multiplicatively complete. The typical argument underlying theorems of this type is illustrated by the case of deficiency 1. Here there exists $f_0(x)$ of norm 1 orthogonal to $\{f_n(x)|n=1, \cdots\}$. Consider $m(x)$, positive, bounded and measurable such that $f_0(x)/m(x)\bar{\epsilon}L_2$. If $g(x)$ is in L_2 and $m(x)g(x)$ is orthogonal to $\{f_n(x)|n=1, \cdots\}$, then $m(x)g(x)=Cf_0(x)$ which implies $g(x)$ cannot be in L_2 unless $C=0$, or $g(x)=0$. The authors present also a procedure for constructing an $m(x)$.

D. G. Bourgin (Princeton, N. J.).

Boas, Ralph P. Quelques généralisations d'un théorème de S. Bernstein sur la dérivée d'un polynome trigonométrique. C. R. Acad. Sci. Paris 227, 618–619 (1948).

The author notes that a theorem of S. B. Stečkin [Doklady Akad. Nauk SSSR (N.S.) 60, 1511–1514 (1948); these Rev. 9, 579], which states that if $T(x)$ is a trigonometric polynomial of degree R and $0<\delta<\pi/R$, then $|T'(x)|\leqq\frac{1}{2}R\csc\delta R\cdot\sup_x|T(x+\delta)-T(x-\delta)|$ is a consequence of a theorem of the reviewer [Duke Math. J. 8, 656–665 (1941); these Rev. 3, 108]. He indicates an extension of the latter theorem so that the quoted result holds for $T(x)$ an entire function of exponential type R [Bernstein, Doklady Akad. Nauk SSSR (N.S.) 60, 1487–1490 (1948); Nikolskiĭ, ibid., 1507–1510 (1948); these Rev. 9, 579].

P. Civin (Eugene, Ore.).

Boas, R. P., Jr. More inequalities for Fourier transforms. Duke Math. J. 15, 105–109 (1948).

Suppose $\phi(t)$ is real and $f(x)=\int_{-\infty}^{\infty}e^{ixt}\phi(t)dt$. Suppose further that $f(x)$ vanishes for $|x|\geqq R$. Then

$$|f(x)|\leqq\tfrac{1}{2}\pi\int_{-\infty}^{\infty}|\phi(t)|dt, \qquad |x|\geqq\tfrac{1}{2}R,$$

and for $\alpha>0$

$$|f(0)|+\alpha|f(x)|\leqq\frac{\pi}{\pi-2\delta}\int_{-\infty}^{\infty}|\phi(t)|dt, \qquad |x|\geqq\tfrac{1}{2}R,$$

where δ is the smallest positive root of $\alpha(\pi-2\delta)=4\sin\delta$. The constants $\frac{1}{2}\pi$ and $\pi/(\pi-2\delta)$ cannot be improved.

The paper also contains a number of corrections to a paper by the author and Kac [same J. 12, 189–206 (1945); these Rev. 6, 265]. The author would like to add the following further correction to that paper: in theorem 8 replace $t>\pi/\beta$ by $t<\pi/\beta$.

A. C. Offord (London).

Duffin, R. J. Function classes invariant under the Fourier transform. Duke Math. J. 15, 781–785 (1948).

Let J_1 be the class of functions defined by

$$\text{(a)}\qquad \left(-x\frac{d}{dx}\right)^n f(x)\geqq0, \qquad 0<x<\infty;\ n=0, 1, \cdots;$$

$$\text{(b)}\qquad f(x)\to0, \quad x\to\infty;$$

$$\text{(c)}\qquad xf(x)\to0, \quad x\to0.$$

Then J_1 is identical with the class of the form

$$f(x)=\int_0^1 x^{-t}d\rho(t), \qquad x>0,$$

where $\rho(t)$ is nondecreasing and continuous at 0 and 1. [For a similar theorem see D. V. Widder, Trans. Amer. Math. Soc. 33, 851–892 (1931), p. 886.] It is proved that if $f(x)$ is in J_1 so are $\int_0^\infty f(t)\sin xtdt$, $(-x\,d/dx)^nf(x)$ and $f(1/x)/x$. Further results of the same nature are established by varying conditions (b) and (c).

H. Pollard.

Doss, Raouf. On the multiplicators of some classes of Fourier transforms. Proc. London Math. Soc. (2) 50, 169–195 (1948).

The author introduces subclasses $\{k\}$ $(k=1, \cdots, 6)$ of $L(-\infty, \infty)$; he then defines (A) $g(u)\epsilon(F_k)$ if it is the Fourier transform of an $f(x)\epsilon\{k\}$, (B) $\lambda(u)$ is a multiplier (j, k) if $g(u)\lambda(u)\epsilon(F_k)$ for all $g(u)\epsilon(F_j)$, and finds necessary and sufficient conditions on $g(u)$ and $\lambda(u)$. In (A), the conditions are expressed in terms of $\sigma_n(x)=\int_{-n}^{n}g(u)(1-|u|/n)e^{iux}du$. For example, $\{2\}$ is the class of essentially bounded functions in $L(-\infty, \infty)$, and $g(u)\epsilon(F_2)$ if and only if (i) $g(u)$ is continuous, (ii) $\int_{-\infty}^{\infty}|\sigma_n(x)|dx\leqq A$, (iii) $|\sigma_n(x)|\leqq B$, with A, B independent of n and x. Some results of this kind have been proved earlier by González Domínguez [Duke Math. J. 6, 246–255 (1940); these Rev. 1, 226]. In (B), the conditions on $\lambda(u)$ are of the form: one of $\lambda(u)$, $\lambda(u)/(u-i)$, $\lambda(u)/(u-i)^2$ shall belong to one of the six classes (F_k). The proofs are analogous to the corresponding ones for Fourier series [for these, see A. Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, chap. IV, and references given there], and also make use of the following theorem of Banach-Steinhaus type. If $\sigma_n(x)\epsilon L(-\infty, \infty)$ and $\limsup_{n\to\infty}\int_{-\infty}^{\infty}|\sigma_n(x)|dx=\infty$, then there exists a uniformly continuous $f(x)\epsilon L(-\infty, \infty)$ such that

$$\limsup_{n\to\infty}\int_{-\infty}^{\infty}dx\left|\int_{-\infty}^{\infty}f(x-y)\sigma_n(y)dy\right|=\infty.$$

G. E. H. Reuter (Manchester).

Sorace, Orazio. Potenze di composizione secondo Volterra della funzione $N(t, s)=\dfrac{a^2}{c}\operatorname{sen}[c(t-s)]$. Matematiche, Catania 2, 65–79 (1947).

Bloch, Pierre Henri. Über den Zusammenhang zwischen den Konvergenzabszissen, der Holomorphie- und der Beschränktheitsabszisse bei der Laplace-Transformation. Comment. Math. Helv. **22** (1949), 34–47 (1948).

Let α, β, γ be the abscissas of absolute, ordinary, and uniform convergence, η the abscissa of holomorphism, μ of boundedness of a Laplace transform. These five quantities satisfy the obvious inequalities $\alpha \geqq \gamma \geqq \beta \geqq \eta$, $\gamma \geqq \mu \geqq \eta$. The author proves that these are the only relations which must hold between the abscissas by showing that to any set of five numbers α, β, γ, η, μ satisfying the inequalities, there exists a Laplace transform having these numbers as corresponding abscissas. The method is analogous to that of L. Neder [Ark. Mat. Astr. Fys. **16**, no. 20 (1922)] who solved the corresponding problem for Dirichlet series. For some of the required examples a suitable form of the Riemann-Lebesgue theorem serves as a basis, but for the examples with $\mu=\eta=-\infty$ certain integrals are used which correspond to the Fejér polynomials employed by Neder. The author restricts himself to Laplace transforms of Riemann integrable functions, but his results evidently also solve the corresponding question for arbitrary Laplace as well as Laplace-Stieltjes transforms. *E. Hille.*

Widder, D. V. A symbolic form of an inversion formula for a Laplace transform. Amer. Math. Monthly **55**, 489–491 (1948).

The author states that the following inversion formula for the Laplace integral $f(s)=\int_0^\infty e^{-st}\varphi(t)dt$, $\varphi(t)$ continuous, can be obtained by methods due to Phragmén and to Fraser [Amer. Math. Monthly **54**, 586–588 (1947); these Rev. **9**, 347]:

$$(*)\qquad \varphi(t)=\lim_{s\to\infty} s\sum_{k=1}^{\infty}\frac{(-)^{k-1}}{(k-1)!}e^{kst}f(ks).$$

[Neither Fraser nor the author takes account of the contribution of Doetsch, Theorie und Anwendung der Laplace-Transformation, Springer, Berlin, 1937, pp. 131, 409.] It is shown that (*) can be symbolized by the operator $1/\Gamma(D)$ after an exponential change of variable. *H. Pollard.*

Titchmarsh, E. C. Complex Fourier-Bessel transforms. Quart. J. Math., Oxford Ser. **19**, 164–176 (1948).

The formula

$$f(x)=\int_0^\infty \mathbf{H}_v(xu)(xu)^{\frac12}du\int_0^\infty Y_v(ut)(ut)^{\frac12}f(t)dt,$$

where Y_v is the Bessel function of second kind, $\mathbf{H}_v$ is Struve's function, and $-\frac12\leqq v<\frac12$, previously established by the author [Proc. London Math. Soc. **22**, xxiv–xxxv (1923)] is proved and is combined with Hankel's formula to give the complex inversion formula

$$f(x)=\tfrac12\int_{ia-\infty}^{ia+\infty}\{J_v(xz)-i\mathbf{H}_v(xz)\}(xz)^{\frac12}F(z)dz,$$

$$F(z)=\tfrac12\int_0^\infty H_v^{(1)}(tz)(tz)^{\frac12}f(t)dt,$$

where $a>0$, $-\frac12\leqq v<\frac12$, $H_v^{(1)}=J_v+iY_v$, $f(t)e^{-at}$ is summable over $(0,\infty)$, and $f(t)$ satisfies certain local conditions. Corresponding formulae are proved for $v\geqq\frac12$; these involve addition of a polynomial in $(tz)^{-1}$ to $H_v^{(1)}(tz)$ in the formula for $F(z)$. The resulting representation for functions defined over $(-\infty,\infty)$ is discussed, and a uniqueness theorem, analogous to that for the author's generalized Fourier transforms, is proved. *J. L. B. Cooper* (London).

Polynomials, Polynomial Approximations

Aparo, Enzo. Sul calcolo delle radici di un'equazione algebrica. Boll. Un. Mat. Ital. (3) **3**, 25–32 (1948).

Let $f(z)=z^n+c_{n-1}z^{n-1}+\cdots+c_0$. Let $f_\nu(z)=z^\nu+c_{n-1}z^{\nu-1}+\cdots+c_\nu$ have the roots α_i of multiplicities μ_i. Let $d_s=\min|\alpha_s-\alpha_i|$ for $i=1,2,\cdots$. Let

$$p_\nu^{n-\nu}=[p_\nu(\lambda)]^{n-\nu}=|c_{\nu-1}|\lambda^{-1}+\cdots+|c_1|\lambda^{-\nu+1}+|c_0|\lambda^{-\nu},$$

where λ is arbitrary. Let $C(a,r)$ be the circle of center a and radius r. Then the author proves the following theorems. (I) Every root of $f(z)$ outside of $C(0,\lambda)$ lies in the domain T for which $|f_\nu(z)|\leqq p_\nu^{n-\nu}$. (II) If $2p_\nu\leqq\min d_s$, then T is contained in the sum of the circles $C(\alpha_s, r_s)$, where r_s is the least positive root of $x^{\mu_s}(d_s-x)^{\nu-\mu_s}=p_\nu^{n-\nu}$. (III) All roots of $f(z)$ lie within the area between $C(0,L_1)$ and $C(0,L_2)$, where L_1 and L_2 are the only positive roots of $x^n=\sum_{h=0}^{n-1}|c_h|x^h$ and $x^n+|c_{n-1}|x^{n-1}+\cdots+|c_1|x=|c_0|$, respectively. Further let $E\varphi(x)/\psi(x)=V(\psi,\varphi,r_1,\cdots,r_i)_{x=-\infty}-V(\psi,\varphi,r_1,\cdots,r_i)_{x=\infty}$, where V denotes the number of variations of sign in the sequence of functions indicated, at the respective arguments, and where $-r_1$ is the remainder of the division ψ/φ, $-r_2$ the remainder of $\varphi/r_1,\cdots$; the last remainder $-r_i$ is a constant. (IV) If $f(z)=u(x,y)+iv(x,y)$ has no root $x+ik$ and has $u(0,0)\neq0$, the number of roots of $f(z)$ having $y>k$ is equal to m, where $2m=n-Ev(x,k)/u(x,k)$. Similarly if $f(z)$ has no root $h+iy$, then the number of roots having $x>h$ is equal to q, where $2q=n+Ev(h,y)/u(h,y)$ for n even or to $2q=n-Eu(h,y)/v(h,y)$ for n odd. *E. Bodewig.*

Bilharz, Herbert. Vereinfachtes Kriterium für Hurwitzsche Gleichungen sechsten Grades. Z. Angew. Math. Mech. **28**, 275–276 (1948).

Referring to the case $n=6$ treated in his previous paper [same Z. **21**, 96–102 (1941); these Rev. **6**, 198], the author gives an elementary geometric method for determining when the vector **C** lies simultaneously in the octant $x<0$, $0<y$, $z<0$ and in the cone $xz-y^2=0$. *M. Marden.*

Kowalewski, Gérard. Remarque sur l'interpolation newtonienne. C. R. Acad. Sci. Paris **227**, 21–23 (1948).

L'écart entre une fonction continue $f(x)$, admettant une dérivée $n$ième continue pour $a\leqq x\leqq b$, et son polynome interpolateur $\sum_{i=1}^n L_i(x)f(x_i)$ avec points x_i $(i=1,\cdots,n; a<x_i<b)$ est donné par:

$$f(x)-\sum_{i=1}^n L_i(x)f(x_i)=\int_a^b\{T(x,u)-\sum_{i=1}^n L_i(x)T(x_i,u)\}f^{(n)}(u)du$$

avec

$$T(x,u)=\tfrac12\frac{(x-u)^{n-1}}{(n-1)!}\operatorname{sgn}(x-u),$$

ou:

$$T(x,u)=\tfrac12\frac{(x-u)^{n-1}}{(n-1)!}[\operatorname{sgn}(x-u)-\operatorname{sgn}(x_0-u)].$$

J. Favard (Paris).

Geronimus, Ya. L. On certain asymptotic properties of polynomials. Mat. Sbornik N.S. **23(65)**, 77–88 (1948). (Russian)

The author gives various conditions on the set of polynomials (1) $P_n(z)=\prod_{i=1}^n[z-z_i^{(n)}]$ that are necessary and sufficient for the quantity $\lim|P_n(z)|^{1/n}$ to exist uniformly in certain sets. His work is based on a theorem by Walsh [Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 20, New York, 1935, § 7.4] to the effect that if C

is a bounded closed set containing all limit points of the set F of zeros of the polynomials (1), and having a connected regular complement, the condition $\lim M_n^{1/n}=d$ is necessary and sufficient for the relation $\lim |P_n(z)|^{1/n}=d|\varphi(z)|$ to hold uniformly on any bounded closed set exterior to C; here $M_n=\max |P_n(z)|$ (z on C), d is the transfinite diameter of the set F, and $\varphi(z)$ is the complex Green's function of the complement of C, with pole at infinity. The principal results are the following. Let $\mathcal{E}=\mathcal{E}_0+\mathcal{E}_1+\mathcal{E}_2$, where $\mathcal{E}_0$ is a discrete denumerable set; where $\mathcal{E}_1$ consists of analytic arcs whose lengths are bounded away from zero; and where $\mathcal{E}_2$ consists of closed regions exterior to $\mathcal{E}_1$, of area bounded away from zero, and with boundaries that consist of a smooth Jordan curve or several smooth Jordan arcs intersecting at nonzero angles. Let C be the boundary of the set $\mathcal{E}_1+\mathcal{E}_2$; let $\sigma(e)$ be a completely additive nonnegative mass function of sets on $\mathcal{E}$; let $m(z_0, \delta)$ denote the mass in the intersection of the circle $|z-z_0|\leqq\delta$ with the set $\mathcal{E}$, and let $a(\delta)=\inf m(z_0, \delta)$ (z_0 on C). Finally, let $P_n(z)$ be the polynomial that minimizes the integral over $\mathcal{E}$, $\int |z^n+\cdots|^k d\sigma$ ($k\geqq 1$). A necessary and sufficient condition for the equation (2) $\lim_{n\to\infty} M_n^{1/n}=d(C)=d(\mathcal{E})$ to hold is that $\lim_{\delta\to 0} \{\delta^{\frac{1}{2}} \log a(\delta)\}=0$. If the mass distribution on the set $\mathcal{E}$ described above is normal, equation (2) is satisfied. *G. Piranian* (Ann Arbor, Mich.).

Leja, F. Sur une propriété des suites de polynomes. Ann. Soc. Polon. Math. 21, 1–6 (1948).

L'auteur démontre par voie élémentaire: si une suite de polynomes $P_n(z)$ (degré $(P_n)\leqq n$) satisfait à $P_n(z)\neq 0$ dans $|z-z_0|<\delta$ et à $|P_n(z)|\geqq M$ sur un continu C contenant z_0, alors à tout $\epsilon\geqq 0$ correspond $\eta\geqq 0$ et un voisinage ω_ϵ de z_0 où l'on a, pour tout n: $|P_n(z)|(1+\epsilon)^n\geqq\eta$.

L'hypothèse de l'auteur faite sur le continu C est inutile. On a simplement: si la suite $P_n(z)$ (degré $(P_n)\leqq n$) satisfait à $|P_n(z_0)|\geqq M$, $P_n(z)\neq 0$ dans $|z-z_0|<\delta$, à $\epsilon>0$ correspond un voisinage ω_ϵ de z_0 où l'on a, pour tout n: $|P_n(z)|(1+\epsilon)^n\geqq M$. Même énoncé en remplaçant $\geqq$ par $\leqq$ et $1+\epsilon$ par $1-\epsilon$: on remarquera en effet que dans $|z-z_0|<\delta/2$ les fonctions $U_n(z)=(1/n)\log |P_n(z)|/|P_n(z_0)|$ sont harmoniques et également continues. *P. Lelong* (Lille).

Leja, F. Sur les polynomes d'interpolation de Lagrange. Ann. Soc. Polon. Math. 21, 80–89 (1948).

The following theorem is proved. Let the boundary F of a domain containing the point at infinity in its interior be expressed as the sum F_1+F_2 of two disjoint nonempty sets (F_1 closed). Suppose for each $n>0$ that the points $\xi_0, \cdots, \xi_n$ belong to F_1+F_2 and define

$$M_n^{(j)}(z)=\max_{\xi_i\in F_j}\prod_{0\leq k\leq n;\, k\neq i}|(z-\xi_k)/(\xi_i-\xi_k)|,\quad j=1, 2;\ i=0, 1, \cdots, n$$

(at least one point ξ_i is assumed to belong to F_1 and $M_n^{(2)}(z)$ is defined as zero if F_2 contains no point of the set $\xi_0, \cdots, \xi_n$). If z_0 is a point of F_2 and if $\liminf_{n\to\infty} [M_n^{(2)}(z_0)]^{1/n}<1$, then $\limsup_{n\to\infty} [M_n^{(1)}(z_0)]^{1/n}>1$. *E. N. Nilson.*

Western, Donald W. Inequalities of the Markoff and Bernstein type for integral norms. Duke Math. J. 15, 839–869 (1948).

Several inequalities are obtained between integral norms of a polynomial in a complex variable and its derivative. The norms used are $\|P\|_C=\{\int_C|P(z)|^p|dz|/\int_C|dz|\}^{1/p}$ and $\|P\|_D=\{\int_D|P(z)|^p dA/\int_D dA\}^{1/p}$, where $P(z)$ is a polynomial of degree n, C is a rectifiable Jordan curve and D is a domain bounded by C. Various hypotheses on the curve C are stated in terms of the function mapping the exterior of C conformally onto the exterior of the unit circle. With $H\geqq 0$ dependent on the particular hypothesis, the results obtained are of the type $\|P'\|_C\leqq An^{1+H}\|P\|_C$. For analytic simple closed curves, the necessity of $H\geqq 0$ is shown. For domains bounded by curves of the latter type, it is shown that $\|P'\|_D\leqq An\|P\|_D$. Further inequalities are given for C the unit circle which relate area norms of $P'(z)$ to curvilinear norms of $P(z)$. The fractional derivative is also treated for C the unit circle or the segment $-1\leqq x\leqq 1$, but in each case the norm of the derivative involves the deletion of a suitable neighborhood of -1. *P. Civin.*

Special Functions

Mikusiński, Jan G.-. Sur les fonctions

$$k_n(x)=\sum_{\nu=0}^{\infty}(-1)^\nu\frac{x^{n+k\nu}}{(n+k\nu)!}$$

($k=1, 2, \cdots$; $n=0, 1, \cdots, k-1$). Ann. Soc. Polon. Math. 21, 46–51 (1948).

The functions $k_n(x)$, $n=0, 1, \cdots, k-1$, form a fundamental system of solutions of the differential equation $z^{(n)}+z=0$, and hence can be expressed in terms of exponential and trigonometric functions. There is also an addition theorem for $k_n(x+y)$. All zeros of any of these functions are simple, and the zeros of two different functions interlace. An estimate for the smallest positive zero of any one of these functions can be given, and the distance of two consecutive large zeros of the same function is approximately $\pi \csc (\pi/k)$. *A. Erdélyi* (Edinburgh).

Bose, B. N. On certain integrals involving Bessel functions. Bull. Calcutta Math. Soc. 40, 8–14 (1948).

The first part of this paper is the continuation of an earlier one [same Bull. 37, 77–80 (1945); these Rev. 7, 441]; the second half lists a few functions (mostly expressible in terms of Bessel functions) which satisfy the integral equation

$$\int_{-\infty}^{\infty}\pi^{-1}(z+t)^{-1}\sin a(z+t)f(t)dt=f(z).$$

A. Erdélyi (Edinburgh).

Beck, Guido. An application of Poisson's integral. Math. Notae 7, 191–204 (1947). (Spanish)

The representation in cylindrical coordinates of the electromagnetic field generated by an electron in uniform rectilinear motion leads to integrals of the type

$$\int_0^\infty\int_0^\infty (cr^2+z^2)^{-\frac{1}{2}}J_0(ar)e^{ibz}r\,dr\,dz.$$

Such integrals are evaluated by using Poisson's integral representations of Bessel functions. *A. Erdélyi.*

Shanker, Hari. Certain integral representations for Whittaker functions. Proc. Cambridge Philos. Soc. 44, 453–455 (1948).

The results presented here are inspired by C. S. Meijer's work: the method used is that of operational calculus. *A. Erdélyi* (Edinburgh).

Harmonic Functions, Potential Theory

Laasonen, Pentti. Über die erste und zweite Randwertaufgabe der präharmonischen und harmonischen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **40**, 28 pp. (1948).

L'auteur expose à nouveau la théorie des fonctions préharmoniques [cf. Laasonen, C. R. Dixième Congrès Math. Scandinaves 1946, pp. 108–117 (1947); ces Rev. **8**, 513], en y ajoutant une solution du "deuxième problème aux limites de la théorie des fonctions harmoniques": détermination d'une fonction harmonique par ses dérivées normales sur le contour. Cette solution est obtenue par passage à la limite à partir de la solution du problème analogue pour les fonctions préharmoniques; la démonstration est précédée de quelques inégalités préparatoires. *H. Cartan* (Paris).

Brelot, M. Sur le principe des singularités positives et la topologie de R. S. Martin. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **23**, 113–138 (1948).

Ce mémoire contient le développement et la démonstration des résultats annoncés dans une note [C. R. Acad. Sci. Paris **226**, 49–51 (1948); ces Rev. **9**, 284] avec quelques raffinements. Nous renvoyons à l'analyse parue ici signalons seulement que l'auteur étudie sur des exemples le comportement de la topologie de Martin; même dans le plan, elle peut décomposer un point frontière, et inversement elle peut accoler des points frontière distincts.

H. Cartan (Paris).

Amerio, Luigi. Sul problema di Dirichlet per l'equazione di Laplace in alcuni campi piani. Pont. Acad. Sci. Comment. **7**, 59–74 (1943).

Let Γ be a simple closed analytic curve. Consider Cauchy data on this curve and the Laplace equation. The solution will, of course, in general be local only. If the normal derivative is prescribed consistently on the curve, then a harmonic solution in the interior of Γ obtains. The author uses this observation to obtain a solution of the Dirichlet problem when Γ is a circle or an ellipse. Thus let τ, $|\tau|=1$, be the complex position parameter on Γ. Let $u(\alpha, z, \bar{z})$ be the analytic function for the Dirichlet data $(\alpha-\tau)^{-1}$, $|\alpha|<1$, on Γ and $v(\alpha, z, \bar{z})$ the corresponding harmonic function when $|\alpha|>1$. These functions may be continued to $|\alpha|=1$ except for a polar singularity. A well-known solution for general Dirichlet data on Γ is represented by a contour integral J, involving $(u-v)|_{|\alpha|=1}$. The point of the paper is the evaluation of u and v. Choose as Cauchy data $(\alpha-\tau)^{-1}$ for the function and a function to be specified presently for the normal derivative. The local solution can be written down explicitly, and it is evident from the explicit formula what the normal derivative must be in order that the solution be actually analytic. The solution in this case is either u or v depending on whether $|\alpha|<1$ or $|\alpha|>1$. In the case of the circle the solution for u and v leads at once to the Poisson integral for J. *D. G. Bourgin* (Princeton, N. J.).

Fichera, G. Teoremi di completezza sulla frontiera di un dominio per taluni sistemi di funzioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 502–507 (1947).

L'auteur considère dans l'espace, par exemple à 3 dimensions, un domaine borné D dont la frontière est formée d'un nombre fini de surfaces assez régulières (Σ_0 enveloppant $\Sigma_1, \cdots, \Sigma_p$). Il introduit $p+1$ points, O_0 quelconque, O_i à l'intérieur de Σ_i, et fait correspondre à chacun un système de fonctions harmoniques du type $Y_m\rho^m$ (pour O_0) ou Y_m/ρ^{m+1} (pour O_i) (Y fonction de Laplace d'ordre m, ρ distance à O_j; à chaque m correspondent $2m+1$ fonctions Y choisis linéairement indépendantes). Il étudie sur la frontière cette famille v_k de fonctions harmoniques et celle des dérivées $\partial v_k/\partial n$. Le principal résultat, seulement annoncé, est que si $f(P)$ est sur la frontière bornée, "quasi-continue," non-négative, mais positive sur un ensemble de mesure superficielle positive, le système des $\partial v_k/\partial n - fv_k$ est "complet au sens de Hilbert," c'est-à-dire que toute suite obtenue par orthogonalisation est complète relativement à la mesure superficielle. [Signalons que le cas particulier des v_k, spécialement mis en évidence, est contenu dans la propriété plus forte antérieurement connue, que le système des v_k est total au sens de Banach (relativement à l'approximation uniforme des fonctions continues), d'ailleurs avec une frontière plus générale; voir Deny, Bull. Soc. Math. France **73**, 71–73 (1945); ces Rev. **7**, 205]. *M. Brelot* (Grenoble).

Fichera, G. Sull'approssimazione delle funzioni armoniche in tre variabili mediante successioni di particolari funzioni armoniche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 508–511 (1947).

L'auteur démontre, en l'étendant beaucoup, et en s'appuyant sur un résultat complétant son article [analysé immédiatement avant celui-ci], un théorème attribué à Nicolesco: une fonction harmonique dans un domaine de frontière connexe assez régulière est limite d'une suite de polynomes harmoniques, uniformement sur tout compact contenu. Ce théorème, contenu pour le plan dans le résultat classique sur l'approximation d'une fonction holomorphe par un polynome, a été établi dans tout espace à n dimensions par adaptation de la méthode de Runge [conférence de Nicolesco à Rome en Novembre 1942; Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 410–423 (1947); ces Rev. **9**, 586]; mais cela est conséquence immédiate d'études antérieures sur l'approximation par fonctions harmoniques d'une fonction continue donnée sur la frontière d'un compact: voir Keldych [C. R. (Doklady) Acad. Sci. URSS (N.S.) **18**, 315–318 (1938)]; et un développement général entraînant aussitôt même les résultats de Fichera dans les articles de Brelot puis Deny [Bull. Soc. Math. France **73**, 55–70, 71–73 (1945); ces Rev. **7**, 205]. Tandis que Brelot comme Nicolesco, mais plus simplement, utilise la méthode de déplacement du pôle de Runge, une solution bien meilleure est donnée par Deny grâce à la notion de système total [voir plus haut]. Fichera se ramenant à un domaine D [voir plus haut] montre l'approximation uniforme sur tout compact contenu, au moyen d'une combinaison linéaire des v_k, de toute fonction finie continue sur $\bar{D}$, harmonique dans D; mais ne disposant directement sur les v_k que d'une propriété trop faible, il passe par l'intermédiaire un peu compliqué de l'approximation en moyenne quadratique sur la frontière de toute fonction de moyenne superficielle nulle au moyen de combinaisons linéaires des $\partial v_k/\partial n$.

M. Brelot (Grenoble).

Verblunsky, S. On a class of harmonic functions. J. London Math. Soc. **23**, 49–56 (1948).

The author writes $u=\delta(G)$ if the harmonic function u, in the domain G, is the difference of two positive harmonic functions. For a point A on the boundary of G, $u=\delta(G; A)$ means that there exists a neighborhood N of A such that $u=\delta(G\cap N)$. The paper is devoted to the proof of the following theorems. (i) If G is divided into G_1, G_2 by a simple

crosscut κ with endpoints at A and B, then $u=\delta(G_1)$, $u=\delta(G_2)$, $u=\delta(G;A)$, $u=\delta(G;B)$ implies $u=\delta(G)$. (ii) If $u=\delta(G,P)$ for every boundary point P of G then $u=\delta(G)$. The paper was evidently written without the knowledge of results obtained by the reviewer [Acta Math. **74**, 65–100 (1941), pp. 68–74; these Rev. **3**, 124] which are closely related to the above theorems. *František Wolf.*

Herriot, John G. Inequalities for the capacity of a lens. Duke Math. J. **15**, 743–753 (1948).

Let ACB and ADB be two coplanar circular arcs bounding a finite area R. If R is revolved about that perpendicular bisector of AB which lies in the plane of R, the resulting solid is called a lens. Let α and β be the angles which BCA and BDA make with AB extended; we assume that α and β are measured in opposite senses, that $\alpha\leqq\beta$, and that $\alpha+\beta\leqq 2\pi$. Then $\alpha+\beta$ is the dielectric angle of the lens. If $\beta\leqq\pi$ and $\alpha+\beta\geqq\pi$, the lens is convex; if $\alpha+\beta=\pi$, it is a sphere; if α/β is constant and $\beta\rightarrow 0$, it approaches two tangent spheres; if $\alpha\rightarrow\pi$ and $\beta\rightarrow\pi$, with AB fixed, it approaches a circular disk.

The author considers the electrostatic capacity of the lens and obtains a number of inequalities relating C to other geometrical quantities. Principal among these are the radius S^* of the sphere which has the same area as the lens, and the outer radius r^* of the meridian section of the lens, namely, the radius of that circle on the exterior of which one can map conformally the exterior of the meridian section, distortionless at infinity. An explicit formula for C exists in the form of a definite integral which is difficult to manage in the general case, but which can be more easily treated in the various special cases. The principal method of the author consists in proving monotonicity properties of C with respect to the various parameters, and deducing inequalities by consideration of the limiting cases. Among the results are the following: $(4/\pi)\log 2\leqq C/r^*\leqq 4/\pi$, equalities holding for equal tangent spheres and circular disk, respectively; C/r^* is greater than or less than one according as $\alpha+\beta$ is greater or less than π; for two tangent spheres, $2^{\frac{1}{2}}\log 2\leqq C/S^*<1$, equality for equal spheres; for the symmetric lens, $2^{\frac{1}{2}}/\pi\leqq C/S^*\leqq 1$, equalities for the disk and sphere, respectively. *J. W. Green* (Los Angeles, Calif.).

Rosenblatt, Alfred. On the gradient of Green's function in the plane. Summa Brasil. Math. **1**, no. 12, 241–246 (1946). (English. Portuguese summary)

The author gives the usual formula for Green's function for a circle, and computes directly the gradient. He then generalizes these formulas for a domain D by making a conformal transformation. Upper and lower bounds for the gradient are obtained; these formulas involve quantities depending on the transformation. *F. W. Perkins.*

Differential Equations

***Sansone, Giovanni. Equazioni Differenziali nel Campo Reale.** Vol. 1, 2d ed. Nicola Zanichelli, Bologna, 1948. xvii+400 pp.

This is the first volume of what is evidently to be a very substantial and valuable treatise on differential equations, with the chief emphasis placed on the real domain. The present volume is devoted exclusively to ordinary differential equations. The contents are divided into six chapters, as follows. (I) Fundamental theorems concerning the existence, uniqueness, and continuity properties of solutions in the real domain. (II) General properties of systems of linear differential equations. (III) Fundamental theorems concerning differential equations in the complex domain, with particular emphasis on linear equations of the second order. (IV) Boundary value problems for an equation of the second order. (V) Boundary value problems for equations of higher orders. (VI) Linear differential equations with periodic coefficients.

Occasionally particular examples are discussed for the sake of illustration, and there are rather extensive sections devoted to some of the linear differential equations which are of importance in mathematical physics. For the most part, however, the attention is focused upon the general theory. The author treats this theory with great care and precision, and the reviewer has found very little to criticize. In some instances, the reasoning adduced in the proofs would have permitted the author to state the theorems in more general and more directly informative forms than the ones which he has chosen. (However, the same criticism would apply to most of the other works on the subject with which the reviewer is familiar.) As can be seen from the foregoing summary of the contents, a large amount of attention is paid to boundary value problems. The treatment of this part of the subject seems to be particularly clear and complete.

A valuable feature of the book is the great number of judicious references to the literature. The detailed table of contents and index make the book very convenient to use for reference. The printing is excellent, and there appear to be only a few trivial typographical errors. *L. A. MacColl* (New York, N. Y.).

Lur'e, A. I. On periodic solutions of systems of linear equations with constant coefficients. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 353–362 (1948). (Russian)

By means of operational methods, the author finds the periodic solution of period T:

$$x_s(t)=\sum_{\lambda=1}^{n}\frac{e^{p_\lambda t}}{1-e^{-p_\lambda T}}\sum_{k=1}^{n}\frac{\Delta_{ks}(p_\lambda)}{\Delta'(p_\lambda)}\times\left\{\int_t^T e^{-p_\lambda\tau}f_k(\tau)d\tau+e^{-p_\lambda T}\int_0^t e^{-p_\lambda\tau}f_k(\tau)d\tau\right\}$$

of the system $\dot{x}_k=\sum a_{ks}x_s+f_k(t)$, where the f_k are also periodic of period T. [There is a printing error in the formula as given in the text.] The p_λ are the (simple) characteristic roots of the equation $\Delta(p)=|a_{ks}-\delta_{ks}p|=0$; $\Delta_{ks}(p)$ is the cofactor of the (k,s) element in this determinant. It is assumed that $e^{-p_\lambda T}\neq 1$; other formulas are derived in cases where this assumption is not satisfied. The case is also considered where the functions f_k and the solutions x_s are periodic of period $2T$ and change signs at the half periods T. *J. L. Massera* (Montevideo).

Blanc, Ch. Sur les équations différentielles linéaires à coefficients lentement variables. Bull. Tech. Suisse Romande **74**, 185–188, 209–213 (1948).

In the first part of this paper the author presents a method for obtaining approximate solutions of linear differential equations with slowly varying coefficients. The method can be described briefly as follows. Consider the equation $Du\equiv a_N(t)u^{(N)}+a_{N-1}(t)u^{(N-1)}+\cdots+a_0(t)u=F(t)$, where the functions $a_n(t)$ are real and bounded, $F(t)$ is bounded, and

the solutions of the corresponding homogeneous equation approach zero (as $t\to\infty$) at least as rapidly as a function $e^{-\alpha t}$. When $F(t)=e^{st}$ we have a solution of the form $u(t)=Y(s,t)e^{st}$; and when $Y(s,t)$ has been determined the solutions in other cases, in which $F(t)$ may have other forms, can be found by familiar methods. To obtain $Y(s,t)$ approximately, the author writes

$$D^*v=c_N(t)v^{(N)}+c_{N-1}(t)v^{(N-1)}+\cdots+c_0(t)v,$$

where D^* is the operator adjoint to D; and he writes

$$J(s,t)=\left\{\sum_{k=0}^{N}(-1)^k c_k(t)s^k\right\}^{-1}.$$

He then concludes, by a somewhat complicated argument, that he can take $Y(s,t)$ as being approximately equal to

$$J(s,t)\left\{1-\frac{\partial}{\partial s}\left(J\frac{\partial}{\partial t}J^{-1}\right)\right\}.$$

The author is cautious about putting forth claims for his method, and he asserts no more than that he considers that the method leads to practically useful results in many cases. The second part of the paper is devoted to an application of the method to the study of a certain electromechanical system. The theoretical results which were obtained have been confirmed by experiment. *L. A. MacColl.*

Wintner, Aurel. The dissipation of internal energy in linear dynamical systems. Philos. Mag. (7) **39**, 722–728 (1948).

This paper is concerned with the stability of a dynamical system for which the Hamiltonian function is a quadratic form in the coordinates and momenta, with coefficients which are functions of the time. The coordinates and momenta are treated as the components of a vector $x(t)$, and the coefficients in the Hamiltonian function are treated as the elements of a matrix $H(t)$. The principal result is contained in the following theorem. On a half-line, say $0\leqq t<\infty$, let $H(t)$ be a $2n$-rowed symmetric matrix of real-valued continuous functions. Suppose that the elements of $H(t)$ are of bounded variation on the half-line, and that the limit matrix $H(\infty)$ (which then necessarily exists) is positive definite. Then $x(t)=O(1)$ as $t\to\infty$ holds for every solution of the dynamical problem. Also the relation $\int^\infty|dh(t)|<\infty$ holds for the energy $h(t)$ of every solution $x=x(t)$. In particular, there exists a finite limit $h(\infty)$ for the energy along every solution. *L. A. MacColl* (New York, N. Y.).

Hartman, Philip, and Wintner, Aurel. On the asymptotic problems of the zeros in wave mechanics. Amer. J. Math. 70, 461–480 (1948).

The following theorem is proved. For $0<s<\infty$ let $f(s)$ be a continuous real function which tends to ∞ as $s\to\infty$. For large s, let $f''(s)$ exist and be continuous, let $f''(s)\geqq 0$, and let $f''(s)=o((f'(s))^{4/3})$. The differential equation $\phi''+(\lambda-f(s))\phi=0$ has, for every real constant λ, exactly one solution $\phi_\lambda(s)$ such that $\phi_\lambda(s)\to 0$ as $s\to\infty$. The largest zero of $\phi_\lambda(s)$ denoted by $s^*(\lambda)$ satisfies the asymptotic formula, as $\lambda\to\infty$, $s^*(\lambda)-f^{-1}(\lambda)\sim\alpha/\{3f'(f^{-1}(\lambda))\}^{1/3}$, where f^{-1} is the inverse function of f and α is an absolute constant; $\alpha=1.85575\cdots$ and is in fact the least positive root of $\int_0^\infty\cos(t^3+rt)dt$. The authors also obtain, under lighter hypotheses, a formula equivalent to Milne's asymptotic formula for the total number of zeros of $\phi_\lambda(s)$ on $0<s<\infty$.
N. Levinson (Copenhagen).

Hartman, Philip, and Wintner, Aurel. On non-conservative linear oscillators of low frequency. Amer. J. Math. **70**, 529–539 (1948).

The equation $x''+f(t)x=0$ is considered with $f(t)$ positive, nonincreasing and continuous for $0\leqq t<\infty$. Let $f(t)\to 0$ as $t\to\infty$. Then: (1) at least one solution is unbounded; (2) there exists a nonoscillatory bounded solution if and only if (*) $\int_0^\infty tf(t)dt<\infty$, in which case there are two solutions $x_1(t)$ and $x_2(t)$ such that $x_1(t)\sim 1$, $x_1'(t)\sim 0$, $x_2(t)\sim t$, $x_2'(t)\sim 1$ as $t\to\infty$; (3) the differential equation can be oscillatory and still have a bounded solution; (4) all solutions are oscillatory and unbounded if $\limsup(-f'/f^{3/2})<4$ and $(-f'/f)'\leqq 0$; (5) all solutions are unbounded if (*) is violated and $-\log f(t)$ is of regular growth. *N. Levinson.*

Rosenblatt, Alfred. On the unicity of solutions of a system of two ordinary differential equations of the first order satisfying given initial conditions in the real domain. Summa Brasil. Math. 1 (1946), no. 13, 247–255 (1948).

The author proves the following theorem. Let the functions $f(x,y,z)$, $g(x,y,z)$ be continuous in the domain D defined by the relations $0\leqq x\leqq a$, $|y|\leqq H$, $|z|\leqq H$. If (x,y_1,z_1), (x,y_2,z_2) are any two points of D with the same coordinate x, let

$$|f(x,y_1,z_1)-f(x,y_2,z_2)|\leqq x^{-m/p}A|z_1-z_2|,$$
$$|g(x,y_1,z_1)-g(x,y_2,z_2)|\leqq x^{-n/p}B|y_1-y_2|,$$

where A and B are constants, and m, n, and p are positive integers such that $(m/p)+(n/p)=2$. Also let there exist constants $\bar{A}$ and $\bar{B}$ such that

$$\bar{A}\geqq A,\quad \bar{B}\geqq B,\quad (n/p)+(n/p)(A/\bar{A})+(B/\bar{B})<1,$$
$$\bar{A}\bar{B}=n/p<1.$$

Then the system of differential equations $y'=f(x,y,z)$, $z'=g(x,y,z)$ has only one solution satisfying the initial conditions $y(0)=z(0)=0$. The reasoning used in the proof is essentially geometrical in character. *L. A. MacColl.*

Rosenblatt, Alfred. On E. Picard's method of successive approximations in the case of a system of two ordinary differential equations of the first order. Summa Brasil. Math. 1 (1946), no. 14, 257–263 (1948). (Spanish)

The author uses a slight variation of the ordinary Picard method of successive approximations to prove the existence and uniqueness of a solution of the system of differential equations $y'=f(x,y,z)$, $z'=g(x,y,z)$ and the initial conditions $y(0)=z(0)=0$, under the following hypotheses: the hypotheses of the theorem stated in the preceding review are retained, except that the sentence beginning "Also let there . . ." is now replaced by "Also let $AB<(m/p)\leqq 2$."
L. A. MacColl (New York, N. Y.).

Cafiero, Federico. Su un problema ai limiti relativo all' equazione $y'=f(x,y,\lambda)$. Giorn. Mat. Battaglini (4) **77**, 145–163 (1947).

G. Zwirner [Rend. Sem. Mat. Univ. Padova **15**, 33–39 (1946); these Rev. **8**, 206] considered the two point boundary problem $y'=\lambda F(x,y(x))$, $y(x_1)=y_1$, $y(x_2)=y_2$, the problem being to determine λ so that the two boundary conditions are satisfied. In the present paper the more general equation $y'=f(x,y(x),\lambda)$, $y(x_1)=\phi_1(\lambda)$, $y(x)=\phi_2(\lambda)$ is treated. The results are too complicated to be summarized briefly. *R. Bellman* (Stanford University, Calif.).

Haag, Jules. Sur certains systèmes d'équations différentielles définies par des fonctions périodiques et discontinues. Bull. Sci. Math. (2) 71, 205–219 (1947).

The problem of existence and stability of periodic solutions for the system $x_i' = \lambda f_i(x_1, \cdots, x_n, t)$ for small λ is considered. The f_i are periodic functions of t, but for a fixed point $(x, \cdots, x_n)$, have a finite number of discontinuities on a period. The results of Poincaré for analytic f_i are carried over by the method of successive approximations. This is an extension of the author's result for continuous f_i [same Bull. (2) 70, 155–172 (1946); these Rev. 9, 92].

P. Hartman (Baltimore, Md.).

Haag, Jules. Sur la synchronisation des systèmes oscillants non linéaires. C. R. Acad. Sci. Paris 227, 649–651 (1948).

This note contains a concise statement of a large number of results concerning the existence and stability of periodic solutions of systems of differential equations. It is stated that the proofs of the theorems will be presented elsewhere. It is difficult to determine just which of the results are to be regarded as being new, and which are old results introduced for the sake of exposition. *L. A. MacColl.*

★Minorsky, N. Modern trends in nonlinear mechanics. Advances in Applied Mechanics, edited by Richard von Mises and Theodore von Kármán, pp. 41–103. Academic Press, Inc., New York, N. Y., 1948. $6.80.

An expository article on the practical aspects of nonlinear differential equations which deals "with the systems with one degree of freedom and is limited to the case when the parameter is a small number." The three chapters are devoted to (1) topological methods including singular points and limit cycles in the Poincaré plane, (2) analytical methods based on small parameter procedures, and (3) nonlinear resonance including subharmonic resonance and entrainment of frequency. A bibliography is appended.

N. Levinson (Copenhagen).

McLachlan, N. W. Periodic solution of a certain nonlinear differential equation. Math. Gaz. 32, 64–66 (1948).

The author discusses periodic solutions of

$$\ddot{y} + ay + by^3 = f \cos \omega t.$$

A first approximation is obtained by a formal power series expansion in $1/\omega$. [The parameter ω is used in a double role, confusing the argument.] *F. Bohnenblust.*

Szarski, Jacek. Sur certains systèmes d'inégalités différentielles aux dérivées partielles du premier ordre. Ann. Soc. Polon. Math. 21, 7–25 (1948).

This paper generalizes the results of M. Nagumo [Jap. J. Math. 15, 51–56 (1938)] and A. Haar [Atti del Congresso Internazionale dei Matematici, Bologna, 1928, v. 3, pp. 5–10 (1930)]. If each of u, v is a function of two sets of variables $x_1, \cdots, x_k; y_1, \cdots, y_n$, represented collectively by $x; y$; if $p_1, \cdots, p_k, q_1, \cdots, q_n; r_1, \cdots, r_k, s_1, \cdots, s_n$ denote their first derivatives; if x_0 are fixed and if (x, y) is in a certain set, then under conditions which take considerable space to state the inequalities $v(x_0, y) < u(x_0, y)$, $f_i(x, y, u, q) < p_i$, $r_i \leq f_i(x, y, v, s)$ imply $v(x, y) < u(x, y)$. A similar result is given for the case where there is only one variable x and there are k each of the unknowns u and v.

J. M. Thomas (Durham, N. C.).

Janet, Maurice. Sur les systèmes comprenant autant d'équations aux dérivées partielles que de fonctions inconnues. C. R. Acad. Sci. Paris 227, 707–709 (1948).

For $i = 1, \cdots, n$ let $E_i = \sum_{j=1}^{n} A_{ij} u_j$, where the u's are functions of the independent variables $(x_1, \cdots, x_{m+1})$ and the A's are linear partial differential operators whose coefficients are holomorphic functions of the x's. The E's are linearly dependent if and only if there exist linear partial differential operators D_i not all of which are zero and for which $\sum_{i=1}^{n} D_i E_i = 0$ identically for all u's. The E's form a normal system if the system got by equating them to zero is equivalent under change of independent variables and algebraic solution to a system solved for the n derivatives $\partial^h u_i / \partial x_1^h$, where $h(i)$ is the order of the system in the unknown u_i. The communication states (i) a set of E's is independent if and only if a normal system can be deduced from it by differentiation and linear combination; (ii) consequently, if from a system S got by equating a set of E's to zero a system T solved for the unknowns is found by differentiation and linear combination, the system S is consistent and T is its solution. *J. M. Thomas* (Durham, N. C.).

★Sommerfeld, Arnold. Partielle Differentialgleichungen der Physik. (Vorlesungen über theoretische Physik, Band VI.) Akademische Verlagsgesellschaft Geest & Portig K.G., Leipzig, 1947. xiii+332 pp. 1st Eng. ed.

The present volume is a slightly enlarged version of a course which forms part of the author's lectures on theoretical physics. In reading this book one should constantly keep in mind that it is not a text-book of mathematical physics, but rather an attempt to motivate mathematical methods in a manner which will appeal to the physicist and convince him (although not the pure mathematician) in a measure which epsilons, deltas and all the paraphernalia of rigorous proofs notoriously fail to achieve. Precisely because this book approaches its subject from the physicist's point of view, rather than duplicating existing mathematical literature, it can be read with pleasure, and with profit, by the mathematician too. It should be especially useful to mathematicians who teach their subject to students of physics.

Chapter I deals with Fourier series and allied topics. Fourier series are introduced through least-square approximations by trigonometric polynomials. Some of their convergence properties (including the Gibbs phenomenon) are briefly mentioned, and Fourier integrals are derived by extending indefinitely the fundamental interval. Series of Legendre polynomials and some other expansions are also mentioned.

Chapter II contains general remarks on partial differential equations, mostly linear partial differential equations of the second order in two independent variables. The introduction of characteristics leads quite naturally to the classification of partial differential equations into elliptic, hyperbolic and parabolic equations. The different nature of these three types, in particular the different boundary conditions natural to them, and the different representations of the solution to which Green's transformation leads in the three cases, are discussed. Adjoint differential operators, Green functions, Riemann's method for hyperbolic equations, and elementary solutions of parabolic equations are also introduced in this chapter.

Chapter III is devoted to problems of heat conduction in one, two, or three (spatial) dimensions. The construction of Green functions by the method of images (including the

third boundary value problem which necessitates the addition to the image proper of a continuous distribution of sources) is the central feature of this chapter, and it serves, among other things, to introduce theta functions and their imaginary transformations. The proof of uniqueness of the solutions of heat conduction problems will probably not fully satisfy the mathematical reader, but it pleasantly rounds off an instructive chapter.

The longest chapter (about one quarter of the whole book) is chapter IV, on the equations of mathematical physics in cylindrical, and spherical polar, coordinates. In it the author develops all properties of Bessel and Legendre functions which are frequently needed by the mathematical physicist.

Bessel functions are introduced through their integral representations which can be interpreted as the generation of cylindrical waves by the superposition of plane waves. [The Bessel function of the first kind is denoted by I_n, although the author mentions the more familiar symbol J_n as being used "in der englischen Literatur" (which, then, would include, among many other well-known German books, "Frank-Mises" with Sommerfeld's own contribution in it). This heterodoxy is the more conspicuous as the author remarks in another footnote in the same chapter: "Überhaupt soll man nicht von dem internationalen Gebrauch abweichen!".] The integral representations are the chief source from which all relevant properties of Bessel functions are derived. Fourier-Bessel expansions are introduced in connection with heat conduction in a cylinder.

Legendre functions are introduced in a corresponding manner through their occurrence in potential theory. Maxwell's generation of spherical surface harmonics as potentials of multipoles, and spherical waves, are also discussed, and Legendre functions of general (noninteger) index, Legendre functions of the second kind, and the hypergeometric function are briefly mentioned. Poisson's formula emerges from an interesting discussion of inversion and its application to potential theory.

Chapter V is an excellent introduction to boundary value problems. Vibrations of circular and rectangular membranes serve to illustrate the occurrence, and some properties, of eigenvalues and eigenfunctions. In this section it is somewhat surprising that the connection with the calculus of variations, and in particular the maximum-minimum properties of eigenvalues, are not mentioned. The discussion of free and forced vibrations leads to the Green function of the wave equation; infinite domains, continuous spectra, and the "radiation condition" are included in the discussion, as is also an application to wave mechanics. Four appendices to this chapter are devoted respectively to the normalisation of eigenfunctions in the continuous spectrum, to a new representation of the Green function of the wave equation for the space outside a sphere, to the separation of the Schrödinger equation in paraboloidal coordinates, and to waves in n dimensions.

Chapter VI discusses the propagation of electromagnetic waves of wireless telegraphy. In comparison with the author's well-known presentation of this subject in "Frank-Mises," we have here a more complete account (for instance, the dipole is situated at an arbitrary distance from the earth, not on its surface), and a more detailed discussion. Throughout this chapter the surface of the earth is regarded as an infinite plane, but an appendix briefly deals with a spherical earth.

A collection of 28 exercises with generous indication of their solutions, and an index, complete this valuable and eminently readable book. *A. Erdélyi* (Edinburgh).

Kendall, David G. A form of wave propagation associated with the equation of heat conduction. Proc. Cambridge Philos. Soc. **44**, 591–594 (1948).

The equation $u_t=h^2u_{xx}+a^2u$ describes diffusion with an exponentially increasing mass. "Wave" solutions $u=f(x-vt)$ are possible only if $|v|\geqq 2ha>0$. This minimum velocity phenomenon is similar to one found by R. A. Fisher in an analogous discrete problem in genetics. For large t the fundamental solution represents approximately a wave with velocity $v_0=2ha$. The same is true if $u(x,0)$ is given by a normal distribution, but it is not true when $u(x,0)=\exp(-|x|)$. *W. Feller* (Ithaca, N. Y.).

Paneth, H. R. Diffusion in a capturing medium of sharply varying geometric cross-section. National Research Council of Canada. Atomic Energy Project. Division of Research. **MP-147** (N.R.C. no. **1585**), i+24 pp. (1 plate) (1945).

The specific problem considered in this paper is that of calculating the distribution of thermal neutrons in a narrow semi-infinite pile of cross section $2a\times 2d$ placed on top of a larger pile of cross section $2b\times 2d$. The corresponding mathematical problem is that of solving the diffusion equation, $\nabla^2\rho=\rho/L^2$, which will be suitable under the following conditions: the solution in the larger pile is assumed to be given by

$$(1)\qquad \rho=\sum_n\sum_k\gamma_{nk}\cos(\pi nx/2b)\cos(\pi ky/2d)$$

at some specified $z=-c$ with known coefficients γ_{nk}. In the smaller pile, extending from $z=0$ to $z=\infty$, the solution must be of the form

$$(2)\qquad \rho=\sum_k\sum_m a_{mk}\cos(\pi mx/2a)\cos(\pi ky/2d)e^{-\nu_{mk}z},$$
$$0\leqq z\leqq\infty;\ -a\leqq x<a;\ -d\leqq y\leqq d,$$

where $\nu_{mk}=[L^{-2}+\frac{1}{4}\pi^2(m^2/a^2+k^2/d^2)]^{\frac{1}{2}}$. At $z=0$ the boundary conditions are that the solution given by (1) and (2) must be continuous for $-a\leqq x\leqq a$ and $-d\leqq y\leqq d$ and that further $\rho=0$ for $+a\leqq x\leqq\pm b$ and $-b\leqq x\leqq -a$. The problem is to determine the coefficients a_{mk} in terms of the coefficients γ_{mk}. It is apparent that the solution of this problem can be obtained by a straightforward procedure. Methods of obtaining approximate relations among the coefficients are considered. *S. Chandrasekhar.*

Walters, A. G. The solution of some transient differential equations by means of Green's functions. Proc. Cambridge Philos. Soc. **45**, 69–80 (1949).

The author relates the Green's function for a steady state equation to that of a time varying equation through the Laplace transform. The steady state equation considered is $D(\phi)=\lambda^2\phi$, where D is a linear differential operator. The time varying equation is $D(\phi)=c^{-2}\phi_{tt}$ or $H^{-1}\phi_t$. With these results the author solves a number of problems in diffusion and wave motion. *A. Heins* (Pittsburgh, Pa.).

Rubinšteĭn, L. I. Concerning the existence of a solution of Stefan's problem. Doklady Akad. Nauk SSSR (N.S.) **62**, 195–198 (1948). (Russian)

In an earlier paper [Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **11**, 37–54 (1947); these Rev. **8**, 516] the author gave a proof for the

existence and uniqueness of the solution of Stefan's problem. The proof given there did not apply to certain special cases. The present paper is an extension of the existence proof for the solution to one of these special cases.
H. P. Thielman (Ames, Iowa).

Traupel, W. Instationäre Wärmeleitungsvorgänge in Platten, Zylindern und Kugeln. Schweiz. Arch. Angew. Wiss. Tech. **14**, 193–205 (1948).

Roettinger, Ida. Note on the use of almost periodic functions in the solution of certain boundary value problems. J. Math. Physics **27**, 232–239 (1948).

The author considers the linear transformation

$$T\{F(x)\}=\int_0^\pi F(x)\phi_n(x)dx=f(k_n),\qquad n=1,2,\cdots,$$

where $\{\phi_n(x)\}=\{A(k_n)\cos k_nx+B(k_n)\sin k_nx\}$ is the sequence of characteristic functions of the Sturm-Liouville problem $y''+k^2y=0$; $L_i(y)=\sum_{j=1}^4 a_{ij}u_j$, $i=1,2$; $u_1=y_1(0)$, $u_2=y'(0)$, $u_3=y(\pi)$, $u_4=y'(\pi)$. The a_{ij}'s are constants. The L_i are taken to be linearly independent here and the k's are assumed to be real. The author discusses this material in the notation of almost periodic functions and gives relations which assist in finding solutions of certain boundary value problems "with intrinsic character," i.e., closed form solutions. Applications are made to problems in heat conduction and the vibrating string. *A. Heins.*

Bourgin, D. G. Two problems of mixed type for the damped wave equation. Quart. Appl. Math. **6**, 279–299 (1948).

This paper deals with the solution of "mixed" problems for the damped wave equation (A) $z_{tt}-z_{xx}+z=0$, which are similar to a problem solved recently by S. Chandrasekhar [Proc. Cambridge Philos. Soc. **42**, 250–260 (1946); these Rev. **8**, 78]: z is to be determined in the unbounded region bounded by a segment $0\leqq x\leqq l$ of the x-axis, and two "timelike" half-lines P and P'. Here P consists of the points of the line $t+\lambda x=0$ with positive t, where $\lambda>1$. In one of the two problems considered the half-line P' is given by $x=l$, $t>0$, in the other problem by $t+\lambda x=\lambda l$, $t>0$. The solution z of (A) to be determined is subject to the boundary conditions $z=z_t=0$ for $0\leqq x\leqq l$, $t=0$; $z_\nu=0$ on P'; $z_\nu=hz+\psi(x,t)$ on P, where h is a positive constant and ψ a given function. Here z_ν denotes the "conormal" derivative of z, formed with respect to the hyperbolic metric with line element $ds=(dt^2-dx^2)^{\frac12}$, which is the metric associated with the characteristic form of the differential equation (A).

The problem of determining z is reduced to the solution of an integral equation for z along P. In obtaining this integral equation from Green's identity extensive use is made of the "method of images," where here, however, the images are not the images formed by ordinary Euclidean reflection, but are taken with respect to the hyperbolic metric mentioned above. The integral equation is solved explicitly in terms of quadratures with the help of Laplace transforms. *F. John* (New York, N. Y.).

Green, G. Problems involving flexural vibrations treated by the wave-train method. Philos. Mag. (7) **39**, 539–546 (1948).

Here boundary conditions at two points are admitted for $\partial^2y/\partial t^2+C\partial^4y/\partial x^4=0$. The basic solution is

$$(1)\qquad y=\exp i(kt-\lambda x),$$

where λ is essentially (α) $\pm(k)^{\frac12}$ or (β) $\pm i(k)^{\frac12}$ and is considered a wave (even in case (β)) that is reflected alternately from one boundary point to the other. A source at $x=x_1$ is (2) $\exp i(kt-\lambda|x-x_1|)$. The sum of the reflections for (2) is computed. One has then the solution for the problem of a periodic force acting at x_1, at least for special boundary conditions. For more complicated sources one uses the Fourier series or integral with k as the frequency parameter. *D. G. Bourgin* (Princeton, N. J.).

Functional Analysis, Ergodic Theory

Ruston, A. F. A note on convexity in Banach spaces. Proc. Cambridge Philos. Soc. **45**, 157–159 (1949).

This note contains the following criteria. A real or complex Banach space B is uniformly convex if and only if for each $\epsilon>0$ there is a $\delta(\epsilon)>0$ such that for x, y in B and f in B^* satisfying $1=\|x\|\geqq\|y\|$, $\|x-y\|\geqq\epsilon$, and $f(x)=\|f\|$, it follows that $\Re(f(y))\leqq\|f\|(1-\delta(\epsilon))$. The space B is strictly convex if and only if for each $f\neq0$ in B^* there is at most one x in B such that $\|x\|=1$ and $f(x)=\|f\|$.
M. M. Day (Princeton, N. J.).

Klee, V. L., Jr. The support property of a convex set in a linear normed space. Duke Math. J. **15**, 767–772 (1948).

The author proves (1) Mazur's theorem [Studia Math. **4**, 70–84 (1933)] that each closed convex set with interior points in a normed space S has a supporting hyperplane at each of its boundary points; (2) if C is a closed convex subset of a normed space S, each of the following conditions implies that C is supported at a dense set of its boundary points: (i) bounded sets in C are weakly compact; (ii) S is a conjugate space and bounded sets in C are weakly compact as sets of functionals; (iii) S is a conjugate space and C is transfinitely closed in S. Result (1) is proved by showing that (3) a convex set C is supported at a point p of C if and only if the cone of half rays from p through C is not dense in S. Result (2) follows from (4): if a point p of C is at minimal distance (in C) from a point of S not in C, then C has a hyperplane of support at p. *M. M. Day.*

Sard, Arthur. Integral representations of remainders. Duke Math. J. **15**, 333–345 (1948).

The unifying theorem of the paper is: if X, Y are Banach spaces and U is linear on X onto Y and V is linear on X onto Z and if $Vx=0$ whenever Ux is, then $Vx=TUx$ for T a linear operator on Y to Z. [The reviewer points out that actually this theorem is just the second isomorphism theorem for topological groups with the evident vector space formulation following the notion of a group with operators, for instance. This observation shows that a similar theorem is valid for vector spaces over suitable division rings. Thus also the author's discussion of completeness of Y as necessary for the validity of the theorem amounts to the requirement of openness for the associated homomorphism since the isomorphism is to be topological.] The author specializes this to obtain neat derivations of various theorems originally published by Peano or Remes. As a typical instance the classic representation of linear functionals in C^n leads to the result: if $V\epsilon(C^n)^*$ and vanishes for polynomials $p(t)$ of degree $n-1$ then $V(x)=\int_0^1 x^n(s)d\alpha(s)$ with $\alpha(s)$ of bounded variation. *D. G. Bourgin.*

Hilding, Sven H. Note on completeness theorems of Paley-Wiener type. Ann. of Math. (2) 49, 953–955 (1948).

Let $\{f_n\}$ be a complete set in Hilbert space H and let $\{g_n\}$ be a set of elements in H satisfying

$$(*)\qquad \|\sum_1^N \alpha_n(f_n-g_n)\| \leqq \lambda[\|\sum_1^N \alpha_n f_n\|^k+\|\sum_1^N \alpha_n g_n\|^k]^{1/k}$$

for every finite sequence $\{\alpha_n\}$. It is shown that if (*) holds for a fixed k and fixed $\lambda<\min[1, 2^{1-1/k}]$ then $\{g_n\}$ is complete. For $k=2$ this reduces to a result of the reviewer [same Ann. (2) 45, 738–739 (1944); these Rev. 6, 127]. Another extension of the reviewer's result is given by Sz. Nagy [Duke Math. J. 14, 975–978; these Rev. 9, 358]. For $k=1$ the reviewer [op. cit.] had the bound $\lambda<2^{-\frac{1}{2}}$; Sz. Nagy [op. cit.], $\lambda\leqq 2^{-\frac{1}{2}}$. The present paper shows that $\lambda<1$ is the true condition. *H. Pollard* (Ithaca, N. Y.).

Kreĭn, M. G., and Krasnosel'skiĭ, M. A. Fundamental theorems on the extension of Hermitian operators and certain of their applications to the theory of orthogonal polynomials and the problem of moments. Uspehi Matem. Nauk (N.S.) 2, no. 3(19), 60–106 (1947). (Russian)

Some useful notions are introduced. By the "inclination" [rastvor] of two linear varieties $\mathfrak{L}_1$, $\mathfrak{L}_2$ in Hilbert space $\mathfrak{H}$ is understood the number

$$\theta(\mathfrak{L}_1, \mathfrak{L}_2)=\max\left[\sup_{f\varepsilon\mathfrak{L}_2, |f|=1}\rho(f, \mathfrak{L}_1), \sup_{f\varepsilon\mathfrak{L}_1, |f|=1}\rho(f, \mathfrak{L}_2)\right],$$

where $\rho(f, \mathfrak{L}_i)=\inf|f-g|$ for $g\varepsilon\mathfrak{L}_i$, $i=1, 2$. We have $\theta(\mathfrak{L}_1, \mathfrak{L}_2)=\theta(\mathfrak{N}_1, \mathfrak{N}_2)$, where $\mathfrak{N}_i$ is the closed subspace orthogonally complementary to $\mathfrak{L}_i$. If $\theta(\mathfrak{L}_1, \mathfrak{L}_2)<1$, then $\mathfrak{N}_1$, $\mathfrak{N}_2$ have equal dimensions. For a linear transformation A with domain $\mathfrak{D}(A)\subset\mathfrak{H}$ the complex number λ is said to be of regular type if $|(A-\lambda I)f|\geqq k_\lambda|f|$ for every $f\varepsilon\mathfrak{D}(A)$ and some $k_\lambda>0$ [see Krasnosel'skiĭ, Doklady Akad. Nauk SSSR (N.S.) 56, 559–561 (1947); these Rev. 9, 242]. The set of all λ of regular type is open. If G is a connected open set composed of numbers of regular type, the closed orthogonal complement to $\mathfrak{R}(A-\lambda I)$ has the same dimension $\mathfrak{c}$ for all $\lambda\varepsilon G$. The cardinal number $\mathfrak{c}$ is called the deficiency index of A in G.

The ideas of M. Riesz [Acta Litt. Sci. Szeged 1, 209–225 (1923)] are generalized to the moment problem $\int_M z^n d\sigma(z)=\mu_n$, where M is a closed set of complex numbers and $\sigma(E)$ a countably additive nonnegative function of Borel subsets of M with $\int_M|z|^n d\sigma(z)<\infty$, and these ideas are translated into the language of the Hilbert space $\mathfrak{L}_\sigma$ composed of all $f(z)$ with $\int_M|f(z)|^2 d\sigma<\infty$, $\int_M|f(z)|d\sigma<\infty$ and with the scalar product $(f, g)=\int_M f\bar{g}d\sigma$. Let $\mathfrak{H}_\sigma$ be the closure, in $\mathfrak{L}_\sigma$, of the set $\mathfrak{P}$ of all polynomials $P(z)$. The orthonormalized sequence $D_0(z)=1$, $D_1(z)$, $D_2(z)$, $\cdots$ of $1, z, z^2, \cdots$, and the transformation $AP(z)=zP(z)$ are considered. The deficiency indices of A are 0 or 1. The sum $\sum_{k=0}^\infty|D_k(\lambda)|^2$ converges or diverges at the same time at all points λ of a component W of the complement of M. Relations between the deficiency index of A in W and of $(z-\lambda)^{-1}$ belonging to $\mathfrak{H}_\sigma$ are studied. If the equality $\int_M(z-\lambda)^{-1}d\omega(z)=0$ for all $\lambda\bar{\varepsilon}M$ implies $\omega(z)=0$ for all complex countably additive $\omega(E)$ defined on Borel subsets of M, the set M is said to satisfy the (α)-condition. This case is studied and applied to the classical Hamburger moment problem: $s_k=\int_{-\infty}^\infty t^k d\sigma(t)$, $k=0, 1, 2, \cdots$. A new proof of the known existence theorem ($\sum_{i,k=0}^n s_{i+k}\xi_i\xi_k$ nonnegative definite for all n) is given by defining a scalar product for polynomials $P(t)=\sum_k\xi_k t^k$, $Q(t)=\sum_k\eta_k t^k$ by the Hankel-bilinear form $(P, Q)=\sum_{i,k}s_{i+k}\xi_i\bar{\eta}_k$ [see J. A. Shohat and J. D. Tamarkin, The Problem of Moments, Math. Surveys no. 2, New York, 1943, p. 1; these Rev. 5, 5]. This leads to some Hilbert subspaces of $\mathfrak{H}_\sigma$. The transformation A is Hermitian. The spectral representation of the resolvent of a self-adjoint extension of A (obtained, if necessary, by extending $\mathfrak{H}$ [see Neumark, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 53–104 (1940); these Rev. 2, 104] immediately furnishes the solution of the moment problem. A criterion, in terms of A, for the existence of a solution is given. There are also new proofs of the classical approximation theorem of M. Riesz [loc. cit.], and of a part of the Nevanlinna theorem. Some results on $(\alpha A+\beta I)(\gamma A+\delta I)^{-1}$ which generalize the topic studied previously by Kreĭn [Rec. Math. [Mat. Sbornik] N.S. 20(62), 431–495 (1947); Mat. Sbornik N.S. 21(63), 365–404 (1947); these Rev. 9, 515] are used as a tool. Kreĭn's theory of extension of generalized Hermitian operators [cf. loc. cit.] is developed again. Applications to the moment problem $\int_{-1}^1 t^k dF(t)=s_k$, to the classical Stieltjes problem, to the trigonometric moment problem and to a generalized one are given. *O. M. Nikodým.*

Hsu, L. C. On Romanoff's device of orthonormalization. Sci. Rep. Nat. Tsing Hua Univ. 5, 1–12 (1948).

[For the relevant papers of N. P. Romanoff, see C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 257–258 (1943); 46, 219–221 (1945); Rec. Math. [Mat. Sbornik] N.S. 16(58), 353–364 (1945); these Rev. 6, 49; 7, 61, 365.] The author extends the results of Romanoff to certain set-theoretic functions defined on a Weisner hierarchy S_∞ [see Trans. Amer. Math. Soc. 38, 474–484 (1935)]. The latter is supposed to be well-ordered and to possess an induced metric, that is, there is a numerically-valued function ϕ on S_∞ such that $\phi(S_m\cap S_n)=(f_m, f_n)$, where f_m belongs to a Hilbert space. Such a ϕ-sequence $\{f_n\}$ can always be orthonormalized by Romanoff's procedure; this involves a Möbius function which is available in a Weisner hierarchy. Conversely, if S_∞ is a hierarchy with a suitably limited multiplication operation, if $\theta(S)\neq 0$ is a multiplicative numerically-valued function on S_∞ and if $\sum[\theta(S_k)]^2<\infty$, then a ϕ-sequence can be formed from any orthonormal system of a Hilbert space. Certain arithmetical identities of Romanoff's remain significant in this new setting. *E. Hille.*

Hsu, L. C. A generalization of Romanoff's method for the construction of orthonormal systems. Acad. Sinica Science Record 2, 178–182 (1948).

Summary of the paper reviewed above.

Watanabe, Satosi. A note on the Dirac space. Progress Theoret. Physics 3, 160–167 (1948).

The author gives some formal properties of operators in a linear space equipped with an invariant Hermitian form which is not positive definite. The paper was written after the author had seen Pauli's article [Rev. Modern Physics 15, 175–207 (1943); these Rev. 5, 277] on Dirac's new method of field quantization but not Dirac's original paper [Proc. Roy. Soc. London. Ser. A. 180, 1–40 (1942); these Rev. 5, 277] and it was intended to supply some of the mathematical details omitted in Pauli's paper.

A. H. Taub (Urbana, Ill.).

Gel'fand, I. M., and Naimark, M. A. Normed rings with involutions and their representations. Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 445–480 (1948). (Russian)

Let R be a normed ring (Banach algebra) with an involution * having the familiar algebraic properties and also satisfying $|x|=|x^*|$. In a previous paper [Rec. Math. [Mat. Sbornik] N.S. 12(54), 197–213 (1943); these Rev. 5, 147] the authors had assumed in addition that $|xx^*|=|x||x^*|$ and $1+xx^*$ has an inverse, and derived an algebraic and norm-preserving isomorphism with an algebra of operators on Hilbert space. The present paper is devoted to exploring in a systematic way the consequences of the weaker axioms. The central notion is the positive functional: a linear functional f satisfying $f(xx^*)\geqq 0$. Functionals which are real on self-adjoint elements are also introduced, and an example shows that they need not be linear combinations of positive functionals. There is a natural correspondence between positive functionals and cyclic representations by operators on Hilbert space (a representation is cyclic if there exists a vector whose transforms are dense). A generalization of Schur's lemma is proved in which the intertwining operator is allowed to be unbounded. A new norm $|x|_1$ is defined by taking the sup of the norms of the operators representing x, and it is shown that $|x|_1$ also equals the supremum of $[f(xx^*)]^{\frac{1}{2}}$, f running over the positive functionals with $f(e)=1$. There may be nonzero elements with $|x|_1=0$; they form a closed ideal I, and the completion R' of R/I is an operator algebra in its usual norm. The ideal I plays a role analogous to a radical, but it is to be noted that I depends not only on the abstract ring R but on its *-operation as well. Raikov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 387–390 (1946); these Rev. 8, 469] had previously studied the case $I=0$.

"Indecomposable" functionals are defined and are shown to be in one-one correspondence with irreducible representations. The Krein-Milman theorem assures one that for any x with $|x|_1\neq 0$ there is an irreducible representation not sending it into 0. The commutative case is next analyzed: R' is then the ring of all continuous functions on the space M of self-adjoint maximal ideals, and the positive functionals are Radon measures on M. If R is the L_1-algebra of a locally compact group G, there is a natural correspondence between representations of R and G. The previous results can then be used to establish the theorem of Gelfand-Raikov on the completeness of the irreducible representations of G. Answering a question implicitly raised by Raikov, they show that the L_1-algebra of the complex unimodular group is not "symmetric," that is, it is not true that xx^* always has a quasi-inverse. Finally it is shown that Godement's formulation of Beurling's theorem [C. R. Acad. Sci. Paris 223, 16–18 (1946); these Rev. 8, 14] is valid in a locally compact group if and only if its L_1-algebra is symmetric.

In a note added in proof, the authors acknowledge that their work overlaps that of Segal [Bull. Amer. Math. Soc. 53, 73–88 (1947); these Rev. 8, 520]. *I. Kaplansky.*

Garcia, Mariano, and Hedlund, Gustav A. The structure of minimal sets. Bull. Amer. Math. Soc. 54, 954–964 (1948).

Let T be an Abelian transformation group acting on a compact metric space X, and let G denote any subgroup of T such that $T=G+C$ for some compact set C. A set $Y\subset X$ is said to be minimal under G if the orbit under G of each point of Y is dense in Y. Assume that X is minimal under T. The minimal sets under G effect a continuous decomposition of X and it is shown that any two of them are equivalent under some transformation belonging to T. It may happen that, for each $\epsilon>0$, G can be chosen so that (i) every minimal set, or (ii) some minimal set, or (iii) the minimal set containing a certain point x, or (iv) the minimal set containing an arbitrary point x, has diameter less than ϵ, or (v) it may happen that X is minimal under every G. Obviously (i)→(iv)→(iii)→(ii). It is shown that (ii)→(iii), that (iv)→(i) in case T is discrete, and that (iii)→(i) in case T is equiuniformly continuous. An example is constructed to show that (iii) does not always imply (iv).

J. C. Oxtoby (Bryn Mawr, Pa.).

Gottschalk, W. H. Transitivity and equicontinuity. Bull. Amer. Math. Soc. 54, 982–984 (1948).

Let G be an Abelian group of homeomorphisms on a compact metric space X, such that the orbit of some point is dense in X. Let H be the group of all homeomorphisms on X that commute with every element of G. Then the following statements are pairwise equivalent: (1) H is transitive in the sense that $xH=X$ for every $x\epsilon X$, (2) H is equicontinuous, (3) G is equicontinuous. This answers a question raised by Hedlund [Amer. J. Math. 66, 605–620 (1944), bottom p. 617; these Rev. 6, 71].

J. C. Oxtoby (Bryn Mawr, Pa.).

Oxtoby, John C. On the ergodic theorem of Hurewicz. Ann. of Math. (2) 49, 872–884 (1948).

A generalized Hurewicz ergodic theorem [for the Hurewicz theorem, see the same Ann. (2) 45, 192–206 (1944); these Rev. 5, 148] is derived by proving that the hypothesis that the given set function be absolutely continuous can be omitted provided that the averaging process is suitably generalized. Let S be a set in which a family $\mathfrak{X}$ of subsets (the measurable sets) is defined such that $S\epsilon\mathfrak{X}$ and differences and countable unions of sets in $\mathfrak{X}$ are in $\mathfrak{X}$. A countably additive set function $F(X)$ defined on $\mathfrak{X}$ is σ-finite if S can be represented as the union of countably many sets $X_n\epsilon\mathfrak{X}$ such that $|F(X_n)|<\infty$. A countably additive set function F is singular with respect to a σ-finite measure μ if F is σ-finite and there exists a set $X_0\epsilon\mathfrak{X}$ such that $\mu(X_0)=0$ and $F(X-X_0)=0$ for every $X\epsilon\mathfrak{X}$. Given a σ-finite measure μ and any countably additive σ-finite set function F on $\mathfrak{X}$, F admits a unique decomposition $F(X)=F'(X)+F''(X)$ into set functions such that F' is absolutely continuous and F'' is singular with respect to μ. It follows from the Radon-Nikodym theorem that corresponding to F' there exists a finite measurable function $f'(x)$ defined on S such that $F'(X)=\int f'(x)d\mu$ for every $X\epsilon\mathfrak{X}$. The function f' is denoted by $D(F;\mu)$. A one-to-one transformation of S onto S is measurable if TX and $T^{-1}(X)$ belong to $\mathfrak{X}$ whenever $X\epsilon\mathfrak{X}$. Given a measurable transformation T, a σ-finite measure μ and a countably additive set function F, the following definitions, like those of Hurewicz, are made:

$$F_n(X)=\sum_{i=0}^{n}F(T^iX),\quad \mu_n(X)=\sum_{i=0}^{n}\mu(T^iX),\quad \varphi_n(x)=D(F_n;\mu_n)$$

and the sequence $\{\varphi_n\}$ is called the averaging sequence of F with respect to T and μ.

The following two theorems are proved. (1) Let T be a measurable transformation, F a countably additive finite set function defined on $\mathfrak{X}$, and μ a σ-finite measure on $\mathfrak{X}$. Then the averaging sequence $\varphi_n(x)=D(F_n;\mu_n)$ converges to a finite limit except on a set N such that $\mu(T^iN)=0$ for

☆ More general results (not assuming the independence of the random variables) were obtained by C. C. Craig [Annual Math. Statistics 7, 1–15 (1936)]; cf. also Aroian [Ibid. 18, 265–271 (1947); these Rev. 9, 48] for asymptotic results for large means. Errata p. 856

every integer $i \geqq 0$. (2) (Generalized Hurewicz theorem.) Let T be a measurable transformation, F a countably additive finite set function defined on $\mathfrak{X}$, and μ a σ-finite measure such that for each $X \varepsilon \mathfrak{X}$ the sum of the series $\sum_{i=0}^{\infty} \mu(T^i X)$ is either 0 or ∞. Then the averaging sequence $\varphi_n(x) = D(F_n; \mu_n)$ converges to a finite limit except on a set N such that $\mu(T^i N) = 0$ for $i = 0, \pm 1, \pm 2, \cdots$, and the limit function $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ has the following properties: (a) $\varphi(x) = \varphi(Tx)$ except on a set N' such that $\mu(T^i N') = 0$ for every integer i, and (b) $\varphi(x)$ is integrable (μ). If in addition $F(X) = 0$ for every set $X \varepsilon \mathfrak{X}$ such that $TX = X$ and $\mu(X) = 0$, then $\varphi(x)$ has the further property: (c) $\int_X \varphi(x) d\mu = F(X)$ for every set $X \varepsilon \mathfrak{X}$ such that $TX = X$ and $\mu(X) < \infty$.

Theorem 2, which is derived first, contains the Hurewicz theorem, but is derived from the Hurewicz theorem, so it represents only a formal generalization. Theorem 1 is proved by decomposing S into a dissipative set S_2, generated by a measurable wandering set, and an invariant set $S_1 = S - S_2$ which contains no wandering set of positive measure, then applying theorem 2 to S_1 and treating S_2, where the situation is essentially simple, separately. Further interesting decompositions of S_1 and S_2 are given. Examples show that the hypotheses of theorem 1 are insufficient to imply the additional conclusion of theorem 2. Finally it is shown that theorem 2 and the Halmos ergodic theorem [Proc. Nat. Acad. Sci. U. S. A. **32**, 156–161 (1946); these Rev. **8**, 34] can each be derived from the other and thus the Halmos and Hurewicz ergodic theorems can be considered equivalent.

G. A. Hedlund (New Haven, Conn.).

Theory of Probability

Hostinský, Bohuslav. On the probability of changes in a system which evolves in the course of time. Rozpravy II. Třídy České Akad. **50**, no. 26, 9 pp. (1940). (Czech)

Let S be a system whose state is characterised by a single real variable x capable of all values in an interval (a, b). At first only continuous changes of S are envisaged, and $j(x, y, s, t)dy$ is the probability that S is in a state between y and $y+dy$ at time t if it is known that it was in state x at time $s<t$. If sudden transitions are introduced into the system and are described by the probability $A(x, y, u)dydu$ that S, in state x at time u, should jump into a state between y and $y+dy$ at some instant between u and $u+du$, the probability distribution is altered. The new probability density of a continuous distribution j_1 satisfies the integral equation

$$j_1(x, y, s, t) = j(x, y, s, t) - \int_s^t \int_a^b \int_a^b j(x, z, s, u) A(z, w, u) j_1(z, y, u, t) dz dw du.$$

This integral equation can be solved by iteration in the usual manner. The probability density for any transition under the new circumstances is given by the infinite series

$$\Phi(x, y, s, t) = j_1(x, y, s, t) + \int_s^t \int_a^b \int_a^b j_1(x, z, s, u) A(z, w, u) j_1(w, y, u, t) dz dw du + \cdots.$$

The author investigates some properties of Φ.

A. Erdélyi (Edinburgh).

Ghizzetti, A. Sul prodotto di due variabili casuali gaussiane. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 534–539 (1948).

The author finds the distribution of the product of two mutually independent Gaussian random variables with unit variance and arbitrary means, in terms of Bessel functions.

J. L. Doob (Urbana, Ill.).

Franchetti, S. Probabilità di errore nelle distribuzioni di Poisson. Pont. Acad. Sci. Comment. **7**, 697–708 (1943).

In section 1, it is shown that the parameter of the Poisson distribution is also the variance. In section 2, the tails are estimated by comparison with a geometric series. In section 3, Stirling's formula is used to estimate the individual terms. Section 4 contains numerical illustrations.

W. Feller (Ithaca, N. Y.).

Seal, H. L. The probability of decrements from a population. A study in discrete random processes. Skand. Aktuarietidskr. **31**, 14–45 (1948).

The decrements mentioned in the title are due to mortality and withdrawals. The main purpose of the paper appears to be to express the parameters of two conditional probabilities in terms of the primary intensity functions. The probabilities considered are given by trinomial and Poisson distributions. A numerical illustration is given for two mortality functions.

W. Feller (Ithaca, N. Y.).

Hoeffding, Wassily, and Robbins, Herbert. The central limit theorem for dependent random variables. Duke Math. J. **15**, 773–780 (1948).

By the Markoff-Bernstein method of grouping, normal tendency is established in a particular case of sums of dependent random variables under conditions on moments and mixed moments.

M. Loève (Berkeley, Calif.).

Borel, Émile. Sur les séquences en météorologie. Annuaire du Bureau des Longitudes **1948**, A.1–A.18 (1948).

Elementary remarks concerning mean lengths and average numbers of success runs in the case of independent trials and of stationary Markov chains. [For applications of the theory of runs to climatology cf. also E. Gold, Quart. J. Roy. Meteor. Soc. **55**, 307–309 (1929); W. G. Cochran, ibid. **64**, 631–634 (1938)].

W. Feller (Ithaca, N. Y.).

Pollaczek, Félix. Sur la probabilité de perte d'un appel téléphonique dans le cas d'un seul groupe de lignes avec blocage temporaire. C. R. Acad. Sci. Paris **226**, 2045–2047 (1948).

In a system of $s \geqq 2$ trunk lines each incoming call during a certain period blocks all lines. Using his method [cf. J. Math. Pures Appl. (9) **25** (1946), 307–334 (1947); these Rev. **9**, 451 and the papers there quoted], the author determines the expected number of lost calls, i.e., of calls arriving at times when all s lines are busy or blocked. The problem contains as a special case that treated by Fortet [same vol., 159–161, 1502–1504 (1948); these Rev. **9**, 361, 518].

W. Feller (Ithaca, N. Y.).

Kampé de Fériet, Joseph. Le tenseur spectral de la turbulence homogène non isotrope dans un fluide incompressible. C. R. Acad. Sci. Paris **227**, 760–761 (1948).

In homogeneous turbulence the velocity and vorticity vectors determine (spatially) homogeneous stochastic processes. The author replaces the usual correlation tensors by

their Fourier transforms, getting the spectral tensors of the two stochastic processes. He finds the general forms of these spectral tensors, and finds each in terms of the other.
J. L. Doob (Urbana, Ill.).

Mathematical Statistics

Uggè, Albino. Di alcune proprietà dei momenti della curva di probabilità e degli indici di normalità. Pont. Acad. Sci. Acta **6**, 229–235 (1942).
The author calculates the absolute moments of the normal distribution and suggests that they might be used for tests of normality. *W. Feller* (Ithaca, N. Y.).

Aitken, A. C. On the estimation of many statistical parameters. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 369–377 (1948).
This paper extends to the multi-parameter situation the attack on the problem of the estimation of statistical parameters which was applied by the author and Silverstone [same Proc. Sect. A. **61**, 186–194 (1941); these Rev. **4**, 25] in the case of a single parameter and by L. Solomon [thesis, University of Edinburgh, 1944] in the case of two parameters. Solomon's results are briefly reviewed with his illustration using the univariate normal frequency function. Again the start is from the two postulates that the estimating functions shall be unbiased and that the determinant of their variance matrix shall be a minimum. A general class of probability density functions admitting this type of estimation is shown to be characterized as solutions of a certain partial differential equation (in matrix form). The results are fully illustrated in the case of a sample from a bivariate normal universe. The general outcome is that now the basic parameters to be estimated are the means and the quadratic moments about the origin. *C. C. Craig.*

Housner, G. W., and Brennan, J. F. The estimation of linear trends. Ann. Math. Statistics **19**, 380–388 (1948).
The problem considered is that of bivariate regression in which both variables are subject to error and have a finite number of means falling on a line and in which the number of sample observations taken about each mean is known. As a means of avoiding the use of unknown weights, in the estimation of the regression coefficient, the authors propose as an estimate the total of the differences of all pairs of observed values of the y's divided by the like total for the observed x's. This estimate is shown to be consistent. The efficiency is examined in the case that both variables are normally distributed about their respective means with the same variance. The asymptotic distribution function is found and it is seen that the asymptotic variance may be quite small. In a particular numerical example it is shown that the efficiency of the proposed estimate compares favorably with that of other estimates which have been recommended. *C. C. Craig* (Ann Arbor, Mich.).

Herbach, Leon H. Bounds for some functions used in sequentially testing the mean of a Poisson distribution. Ann. Math. Statistics **19**, 400–405 (1948).
Let z be the logarithm of the likelihood ration for a single observation for testing θ_1 against θ_0 values of the parameter of a Poisson distribution. Let α and β be the sizes of the errors of the first and second kind, $L(\theta)$ the operating characteristic, h the nonzero root of $E(e^{hz})=1$, $A=(1-\beta)/\alpha$, and $B=\beta/(1-\alpha)$. Bounds on $L(\theta)$ are, when $h>0$, $(1-A^h)/(\eta B^h-A^h)\leqq L(0)\leqq(1-\delta A^h)/(B^h-\delta A^h)$, and δ and η are interchanged when $h<0$; here $\eta=\inf_\zeta E(e^{hz})$ for $e^{hz}<1/\zeta$, $\zeta>1$; $\delta=\sup_\zeta E(e^{hz})$ for $e^{hz}>1/\zeta$, $0<\zeta<1$. Corresponding bounds on the expected sample size are given.
A. M. Mood (Santa Monica, Calif.).

Wald, A., and Wolfowitz, J. Optimum character of the sequential probability ratio test. Ann. Math. Statistics **19**, 326–339 (1948).
It is shown that the sequential probability ratio test for testing a null hypothesis against a single alternative has a smaller average sample size than any other test with the same or more stringent control of the errors. The proof consists of constructing an average risk function which is linear in the expected sample size, showing that the ratio test minimizes the risk, then using the inequality on the risks to get the desired inequality on the expected sample sizes.
A. M. Mood (Santa Monica, Calif.).

Banerjee, K. S. Weighing designs and balanced incomplete blocks. Ann. Math. Statistics **19**, 394–399 (1948).
The weighing problem leads to a system of regression equations $W_\alpha=\sum_1^m x_{\alpha i}w_i+\epsilon_\alpha$ ($\alpha=1, \cdots, n$), where W_α is the weight in the left pan at the αth weighing, w_i the weight of the ith object, $x_{\alpha i}=1$ if the ith object is in the right pan at the ith weighing, $x_{\alpha i}=-1$ if it is in the left pan, $x_{\alpha i}=0$ if it is in neither and the ϵ_α are independently distributed with mean 0 and common variance σ^2. The resulting least squares equations may be written in the form $WX=wX'X$, where $W=(W_1, \cdots, W_n)$, $w=(w_1, \cdots, w_m)$, $X=(X_{\alpha i})$. The covariance matrix of the least squares estimates of the w_α is then given by $\sigma^2(X'X)^{-1}$. The author discusses various designs (choices of the $x_{\alpha i}$), in particular, incomplete balanced block designs, where the objects weighed correspond to the varieties of an incomplete balanced block design and in the ith weighing the objects in the ith block are all put in one pan. The resulting normal equations are then the normal equations for the interblock estimates of an incomplete balanced block design. The author also discusses the estimation of the weight of linear combinations of the objects, in particular their mean. *H. B. Mann.*

Bose, R. C. Mathematical theory of the symmetrical factorial design. Sankhyā **8**, 107–166 (1947).
This is a comprehensive expository treatment of the application of finite geometries to the theory of confounding in factorial designs. In § 1 the concepts of treatment contrast, main effect and interaction are defined and it is shown that for an s^m design (m factors each at s levels) it is possible to split the s^m-1 degrees of freedom completely into $(s^m-1)/(s-1)$ sets of $s-1$ each, if s is a power of a prime. This is achieved as follows: the treatment where the ith factor is at the level x_i is mapped into the point $(x_1, \cdots, x_m)$ of E.G.(m, s). Consider pencils $P(a_1, \cdots, a_m)$ of $(m-1)$-flats in E.G.(m, s) given by all hyperplanes $a_0+a_1x_1+\cdots+a_mx_m=0$, where a_0 varies over all s values of G.F.(p^n) and $a_1, \cdots, a_m$ are kept fixed. The differences between the treatments occurring in the hyperplanes yield then the $s-1$ degrees of freedom carried by the pencil $P(a_1, \cdots, a_m)$. These belong to the interaction between the factors $i_1, \cdots, i_k$ if $a_{i_1}\neq0, \cdots, a_{i_k}\neq0$, $a_j=0$ for $j\neq i_t$.
The construction of confounded designs is discussed in § 2. To construct designs of s^k blocks of s^{m-k} treatments

each in which the degrees of freedom carried by the independent pencils $P_1, \cdots, P_k$ are confounded we choose an $(m-1)$-flat from each of the pencils P_i. The intersection of all these $(m-1)$-flats is an $(m-k)$-flat. Taking all possible choices one obtains a total number s^k of $(m-k)$-flats. The treatment corresponding to the points of one $(m-k)$-flat are then assigned to one block. Thus s^k blocks are obtained in which the degrees of freedom carried by the pencils $P_1, \cdots, P_k$ and all their linear combinations are confounded. One may also consider the degrees of freedom carried by these pencils as the main effects and interactions of k generalized factors.

Partial confounding and the problem of balancing of partially confounded designs are discussed in § 3. If an experiment is replicated a number of times it is of advantage to confound different interactions in different replications. If ith order interactions are confounded the problem arises of constructing designs which confound each ith order interaction equally often. The author gives solutions to the problem of balancing in various special cases.

In the last paragraph he considers the problem of constructing designs in which interactions of a given order, say t, and lower remain unconfounded. The maximum number $m_t(r, s)$ of factors in a symmetrical factorial design where each factor is at s levels and each block contains s^r treatments, while interactions of order t and lower remain unconfounded, is equal to the maximum number of points in P.G.(r, s) such that any t of them span a P.G.$(t-1, s)$. The following results are derived: $m_2(r, s) = (s^r-1)/(s-1)$ [due to R. A. Fisher, Ann. Eugenics **11**, 341–353 (1942); **12**, 283–290 (1945); these Rev. **4**, 127; **7**, 107]; $m_3(3, s) = s+2$ (if s is a power of 2), $= s+1$ (if s is the power of an odd prime); $m_3(4, s) = s^2+1$ if s is a power of an odd prime; $m_3(r, s) = 2^{r-1}$ if $s=2$. *H. B. Mann* (Columbus, Ohio).

Nandi, H. K. A mathematical set-up leading to analysis of a class of designs. Sankhyā **8**, 172–176 (1947).

The author considers pqr-variate distributions x_{ijk}, $i=1, \cdots, p$; $j=1, \cdots, q$; $k=1, \cdots, r$, where $E(x_{ijk}) = m_{ijk}$, $\sigma(x_{ijk}) = \sigma$, $\sigma(x_{ijk}x_{i'j'k'}) = \sigma_1^2$ if $i=i'$, $j=j'$, $k \neq k'$; $= \sigma_2^2$ if $i=i'$, $j \neq j'$; $= \sigma_3^2$ if $i \neq i'$. Supposing that the m_{ijk} are linear functions of unknown parameters the author discusses the problem of estimating these parameters and of testing hypotheses about them. The analysis of split plot and strip arrangements, the intra- and interblock estimates and the intraclass correlation coefficient are special cases of the general theory discussed by the author. *H. B. Mann.*

Harshbarger, Boyd. Rectangular lattices. Virginia Agricultural Experiment Station, Memoir **1**, iii+26 pp. (1947).

Harshbarger, Boyd. Preliminary report on the rectangular lattices. Biometrics **2**, 115–119 (1946).

The design proposed by the author can best be obtained by writing out a square lattice and then deleting varieties $v_{12}, \cdots, v_{k-1\,k}, v_{k1}$. The author derives completely the analysis of this type of design using also the interblock information. *H. B. Mann* (Columbus, Ohio).

TOPOLOGY

de Rham, Georges. Remarque au sujet de la théorie des formes différentielles harmoniques. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **23**, 55–56 (1948).

The author notes that one of the theorems given by him in a former paper [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **22**, 135–152 (1946); these Rev. **8**, 603], namely, that an analytic manifold with an analytic ds^2 can be imbedded analytically in Euclidean space, had been proved by S. Bochner [Duke Math. J. **3**, 339–354 (1937)].
H. Whitney (Cambridge, Mass.).

Dubois-Violette, Mme. Pierre-Louis. Sur les cycles limites des réseaux de courbe couvrant une surface de genre $p>1$. C. R. Acad. Sci. Paris **226**, 1068–1069 (1948).

Dubois-Violette, Mme. Pierre-Louis. Sur les réseaux de courbes couvrant une surface de genre $p>1$ et n'admettant aucun cycle limite. C. R. Acad. Sci. Paris **226**, 1580–1582 (1948).

Dubois-Violette, Mme. Pierre-Louis. Sur les réseaux de courbes sans cycle limite, tracés sur une surface de genre p. C. R. Acad. Sci. Paris **226**, 1676–1678 (1948).

In the first note, families of curves on a surface R with only a simple type of singularity are studied. One result: the maximum number of topologically distinct limit cycles is $3p-1$ if the genus p satisfies $p>1$, 1 if $p=1$, 0 if $p=0$. In the other two notes, it is assumed that the family contains no closed curve. If some curve has every point of R as accumulation point, so do all curves of the family. Such families exist on a surface of arbitrary genus. [Proof. Take such a family on a torus. Cut it along a short cross section, and attach a similar torus to it, obtaining a family on a surface of higher genus, etc.] The results are not easy to understand, because of the sketchy nature of the definitions and proofs. *H. Whitney* (Cambridge, Mass.).

Scorza Dragoni, G. A proposito di un teorema fondamentale sulle traslazioni piane generalizzate: considerazioni preliminari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 470–474 (1947).

Scorza Dragoni, G. A proposito di un teorema fondamentale sulle traslazioni piane generalizzate: proposizioni preliminari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 474–478 (1947).

Scorza Dragoni, G. A proposito di un teorema fondamentale sulle traslazioni piane generalizzate: dimostrazione nel caso di vertici di prima categoria. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 50–53 (1948).

Scorza Dragoni, G. A proposito di un teorema fondamentale sulle traslazioni piane generalizzate: compimento della dimostrazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 180–183 (1948).

Scorza Dragoni, G. Un contributo ulteriore ad un teorema sulle traslazioni piane generalizzate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 284–289 (1948).

Let t be a sense-preserving fixed-point-free topological automorphism of the plane. A simple arc λ is a translation arc if λ and $t(\lambda)$ have a common end-point but otherwise fail to intersect. Let $\sigma(\lambda)$ be the trajectory of t generated by λ and let $\Sigma(\lambda)$ be one of the two components of the complementary set of $\sigma(\lambda)$. The author considers only translation arcs which are polygonal lines, all corners being right angles. The five notes are largely devoted to establishing the proposition that λ always contains fundamental or meta-

fundamental points relative to $\Sigma(\lambda)$. A point P of λ is fundamental if P is the origin of a ray r which lies (except for P) in $\Sigma(\lambda)$, is perpendicular at P to a side of λ, and which fails to intersect its image under t; or if P is an end-point of a segment $s=PQ$ which is perpendicular at P to a side of λ, lies (except for P) in $\Sigma(\lambda)$, and is such that $t(Q)$ is exactly the intersection of s with $t(s)$ or with $t^{-1}(s)$. The definition of metafundamental points results from replacing r and s respectively by the union of a segment and a ray perpendicular to the segment, and by the union of two perpendicular segments. *P. A. Smith.*

Wu, Wen-tsun. On the product of sphere bundles and the duality theorem modulo two. Ann. of Math. (2) **49**, 641–653 (1948).

Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be sphere bundles over complexes K_1 and K_2. Then their product bundle $\mathfrak{S}$ is defined in a natural way over the product complex $K_1\times K_2$. If $K_1=K_2=K$, the part of $\mathfrak{S}$ over the diagonal $K\times K$ is the product as defined by the reviewer [Lectures in Topology, pp. 101–141, University of Michigan Press, Ann Arbor, Mich., 1941; these Rev. **3**, 133]. Certain canonical sets of vector fields are defined in the $\mathfrak{S}_i$; these define the characteristic classes W_1, W_2. Combining these in a certain way gives sets of vector fields in $\mathfrak{S}$, defining the W^r. It is shown that W^r is the product cochain $\sum_{i=0}^r W_1{}^i\times W_2{}^{r-i}$. From this the duality theorem (mod 2) of the reviewer [loc. cit.] is deduced. [This proof is incorrect, for the mapping $\sigma\rightarrow\sigma\times\sigma$ is not a chain transformation; but the fact is true.] This proof of the duality theorem is considerably simpler than that of the reviewer [unpublished]. *H. Whitney.*

Wu, Wen Tsun. Sur l'existence d'un champ d'éléments de contact ou d'une structure complexe sur une sphère. C. R. Acad. Sci. Paris **226**, 2117–2119 (1948).

If a sphere has a continuous family of tangent p-planes, then it has a continuous family of sets of p independent tangent vectors. [A somewhat more general theorem is given; the proof is simple.] Hence for the various cases in which families of the second type are known not to exist, families of the first type do not exist either. Spheres S^{4r} do not admit a complex structure (or even an "almost complex structure"). *H. Whitney* (Cambridge, Mass.).

Wu, Wen-tsun. Sur les classes caractéristiques d'un espace fibré en sphères. C. R. Acad. Sci. Paris **227**, 582–584 (1948).

Let S be a sphere bundle over a differentiable manifold M. A theorem is given relating the characteristic classes of S and of the tangent bundles of M and of S. [The proof is not clear to the reviewer.] The following consequences are deduced. If M can be fibered into d-spheres, then the characteristic classes of M vanish in dimensions $n-d$. All the characteristic classes of the Stiefel manifolds (sets of unit orthogonal vectors in a Euclidean space) vanish. *H. Whitney* (Cambridge, Mass.).

Lichnerowicz, André. Un théorème sur l'homologie dans les espaces fibrés. C. R. Acad. Sci. Paris **227**, 711–712 (1948).

Let E_{n+q} be a fiber space with base space B_n and fiber F_q; it is supposed that all are compact orientable manifolds, and that the fibers are not ~ 0 in E. Each chain C_k in B defines a chain C_{k+q} in E; this gives a homomorphism of the $(k+q)$th cohomology group of E onto the qth of B. The Betti numbers b_k satisfy $b_{k+q}(E)\geqq b_k(B)$ $(k=0,\cdots,n)$. The proof uses differential forms and the de Rham theorem. (Using topological methods, the assumption that the spaces are manifolds may be omitted, if the fibers are cycles, each two homologous.) Suppose the fibers are spheres S_q, $q\geqq n-1$. Then using inequalities of Leray [same C. R. **223**, 412–415 (1946); these Rev. **8**, 166], it follows that $b_k(E)=b_k(B\times S_q)$. *H. Whitney* (Cambridge, Mass.).

Eilenberg, Samuel. Relations between cohomology groups in a complex. Comment. Math. Helv. **21**, 302–320 (1948).

L'auteur se place dans un complexe simplicial K. En rapprochant la théorie de la cohomologie des groupes [Eilenberg-MacLane, Ann. of Math. (2) **46**, 480–509 (1945); ces Rev. **7**, 137] et la théorie de la cohomologie simpliciale avec des "systèmes locaux" de coefficients [Steenrod, Ann. of Math. (2) **44**, 610–627 (1943); ces Rev. **5**, 104], il est conduit à de nouveaux cocycles caractéristiques d'un complexe K.

Un groupe Q (non nécessairement abélien), un groupe abélien G et un homomorphisme f de Q dans le groupe Aut (G) des automorphismes de G, définissent des groupes de cohomologie $H^n(Q,G)$. Le groupe $H^2(Q,G)$ s'interprète comme "groupe d'extensions" de Q (quotient) par G (sous-groupe) relativement à f. Si $Q=F/R$, Q opère sur R_0, quotient de R par le sous-groupe $[R,R]$ des commutateurs; d'où un élément $\alpha\varepsilon H^2(Q,R_0)$. Si en outre Q opère sur G, Q opère sur $G^*=\text{Hom}\,(R,G)=\text{Hom}\,(R_0,G)$; d'où $H^n(Q,G^*)$. En multipliant par α on obtient un homomorphisme $H^n(Q,G^*)\rightarrow H^{n+2}(Q,G)$; si F est libre, c'est un isomorphisme sur, pour tout $n\geqq 1$ [Eilenberg-MacLane, loc. cit.].

L'auteur donne un exposé complet de la théorie des coefficients locaux dans un complexe simplicial connexe K. Pour tout G abélien, on sait que tout homomorphisme du groupe fondamental $\pi_1(K)$ dans Aut (G) définit un système local, noté $\bar{G}$, de groupes isomorphes à G; tout système local est ainsi obtenu une fois et une seule. L'auteur définit [cf., pour le cas de l'homologie singulière d'un espace topologique, Eilenberg, Trans. Amer. Math. Soc. **61**, 378–417 (1947); **62**, 548 (1947); ces Rev. **9**, 52] des homomorphismes canoniques $H^n(\pi_1(K),G)\rightarrow\Lambda^n(K,\bar{G})\rightarrow H^n(K,\bar{G})$, où $\Lambda^n(K,\bar{G})$ est le groupe des classes de cohomologie qui sont des "annulateurs sphériques." Un théorème d'Eckmann-Eilenberg-MacLane, formulé pour les coefficients locaux, dit: si $\pi_n(K)=0$ pour $1<n<q$, alors $H^n(\pi_1(K),G)\rightarrow\Lambda^n(K,\bar{G})$ est un isomorphisme sur, pour $n\leqq q$. En particulier, $H^2(\pi_1(K),G)$ s'identifie toujours à $\Lambda^2(K,\bar{G})$. Or $\pi_1(K)$ est quotient du groupe libre $\pi_1(K^1)$ (K^1 squelette de dimension un) par un sous-groupe χ_1; soit $\theta_1=\chi_1/[\chi_1,\chi_1]$; $\pi_1(K)$ opère sur θ_1; d'où un élément de $H^2(\pi_1(K),\bar{\theta}_1)$ et par suite un élément $\xi^2\varepsilon H^2(K,\bar{\theta}_1)$. L'auteur donne une définition directe de ξ^2, en associant à chaque 2-simplexe ordonné de K l'élément de $\bar{\theta}_1$ qui lui correspond naturellement. Pour tout système local $\bar{G}$, soit $\bar{G}^*=\text{Hom}\,(\bar{\theta}_1,\bar{G})$; la multiplication par ξ^2 définit un homomorphisme: $H^n(K,\bar{G}^*)\rightarrow H^{n+2}(K,\bar{G})$; c'est un isomorphisme sur, pour $1\leqq n\leqq q$, si $\pi_n(K)=0$ pour $1<n\leqq q+1$.

La définition de ξ^2 est généralisée: pour toute dimension $n\geqq 2$, l'auteur définit un $\xi^{n+1}\varepsilon H^{n+1}(K,\bar{\theta}_n)$, $\bar{\theta}_n$ désignant le système local relatif au sous-groupe du groupe d'homotopie π_n du squelette K^n, formé des éléments dont l'image dans $\pi_n(K)$ est nulle. *H. Cartan* (Paris).

Hu, Sze-tsen. On the homology sequence of an abstract complex. J. London Math. Soc. **22** (1947), 275–281 (1948).

Three theorems on the decomposition of homology groups into direct sums are obtained as consequences of the exact-

ness of the homology sequence of a complex and subcomplex. *N. E. Steenrod* (Princeton, N. J.).

Hu, Sze-tsen. Homotopy groups of some mapping spaces of spheres. Bull. Calcutta Math. Soc. 39, 127–130 (1947).

Let Y, Y_0 be connected, locally-contractible, separable metric spaces, and $Y_0 \subset Y$. Let Λ^m be the space of maps of an m-sphere into Y carrying a reference point x_1 into Y_0. Let Ω^m be the subspace of maps which carry a second reference point x_0 into a point $y_0 \epsilon Y_0$. For $r>1$, it is shown that the homotopy group $\pi_r(\Lambda^m)$ $[\pi_r(\Omega^m)]$ is naturally isomorphic to the direct sum $\pi_{m+r}(Y)+\pi_r(Y_0)$ $[\pi_{m+r}(Y)+\pi_{r+1}(Y, Y_0)]$. A more involved result is given for $r=1$. This extends previous results of the author [Ann. of Math. (2) 48, 717–734 (1947); these Rev. 9, 197]. *N. E. Steenrod.*

Hu, Sze-tsen. On Lipschitz mappings. Portugaliae Math. 7, 45–49 (1948).

A mapping f of one metric space into another is "Lipschitz" if, for some N, $\rho[f(x), f(y)] \leqq N\rho(x, y)$. Some elementary properties are proved. The proof in (2.1) is incorrect; the hypothesis $m>2n$ in (2.4) is unnecessary. The Lipschitz singular homology groups and the Čech homology groups of a "Lipschitz space" are isomorphic. *H. Whitney* (Cambridge, Mass.).

Stone, A. H. Paracompactness and product spaces. Bull. Amer. Math. Soc. 54, 977–982 (1948).

L'auteur résout un problème posé par le rapporteur en montrant que tout espace métrisable est paracompact. Plus généralement il prouve que les espaces paracompacts sont identiques aux espaces pleinement normaux de Tukey [Convergence and Uniformity in Topology, Princeton University Press, 1940; ces Rev. 2, 67], c'est-à-dire aux espaces séparés E ayant la propriété suivante: pour tout recouvrement ouvert (V_α) de E, il existe un recouvrement ouvert (W_β) de E tel que pour tout $x \epsilon E$, la réunion des W_β tels que $x \epsilon W_\beta$ soit contenue dans un V_α. Il est facile de voir que tout espace paracompact est pleinement normal; la réciproque résulte d'une construction d'un recouvrement ouvert localement fini, à l'aide d'une double récurrence. Tout espace métrique étant pleinement normal d'après Tukey [loc. cit.] est donc paracompact. Dans une seconde partie, l'auteur montre que le produit d'une infinité non dénombrable d'espaces isomorphes à l'espace discret N des entiers naturels n'est pas normal; en utilisant ce résultat et sa caractérisation des espaces paracompacts, il montre que pour qu'un produit d'espaces métriques soit normal, il faut et il suffit qu'il y ait au plus une infinité dénombrable d'espaces facteurs non compacts. *J. Dieudonné* (Nancy).

Morita, Kiiti. Star-finite coverings and the star-finite property. Math. Japonicae 1, 60–68 (1948).

A space has the star-finite property if every open covering has some star-finite refinement, where a covering is star-finite if each set of it meets only finitely many of its fellows [cf. J. W. Tukey, Convergence and Uniformity in Topology, Princeton University Press, 1940; these Rev. 2, 67]. First the author proves some results about Δ-refinements [see Tukey], and about normal spaces and normal coverings. Among the latter are included Tukey's V-5.4 and V-6.1, of which the original proofs are considered to be subject to some question because of the obscurity of Tukey's V-2.5. The main purpose of the paper is to prove that a regular space with the star-finite property is fully normal [Tukey's term]. A result stronger than this was independently obtained recently by A. H. Stone [cf. the preceding review], who established the equivalence of full normality and paracompactness for T_1-spaces. The star-finite property implies paracompactness, of course. The present paper closes with applications to locally compact Hausdorff spaces R, such as proving that R is fully normal if and only if it has the star-finite property. *R. Arens* (Los Angeles, Calif.).

Begle, Edward G. Topological groups and generalized manifolds. Bull. Amer. Math. Soc. 54, 969–976 (1948).

The principal results of this paper are contained in the following theorems. (I) Let G be a locally compact space which is both a topological group and an n-dimensional orientable generalized manifold. Let H be a closed connected $(n-1)$-dimensional subgroup. Then, if H carries a nonbounding $(n-1)$-cycle, H is also an orientable generalized manifold. (II) Let G be a locally compact separable metric topological group which is also an orientable n-dimensional generalized manifold. Let H be a closed connected $(n-1)$-dimensional subgroup. Then H is an orientable generalized manifold if any one of the following conditions is satisfied: (1) H separates some open set of G; (2) for some open set O of H, there is a nonbounding $(n-1)$-cycle of $H \bmod H-O$; (3) G is locally Euclidean. These theorems generalize a result of the reviewer [Ann. of Math. (2) 49, 118–131 (1948), p. 122, lemma 2.1; these Rev. 9, 496] to the effect that if H is a closed connected two-dimensional subgroup of a locally Euclidean 3-dimensional group then H is a two-dimensional manifold. *D. Montgomery.*

GEOMETRY

Hjelmslev, Johannes. Contributions to descriptive curve theory. Mat. Tidsskr. B. 1948, 1–23 (1948). (Danish)

The first part of the paper considers plane curves which consist of a finite number of circular arcs (straight segments included) such that two such arcs have a common tangent, cusps admitted, at their joining point. A closed curve without cusps is called ordinary. Typical results are as follows. An ordinary curve consisting of 4 circular arcs is cut in at most 4 points by any circle which does not contain one of the arcs. The joining points of an ordinary curve with an odd number of arcs can be prescribed arbitrarily. If the number of arcs is $2n$, then $2n-1$ joining points can be chosen arbitrarily, but the last must lie on a circle determined by the $2n-1$ first.

The second part considers similar problems for spatial curves composed of a finite number of arcs of helices. For two arbitrary line elements (x, dx) and (y, dy) there is a helicoidal arc connecting (x, dx) to (y, dy) if and only if dx and dy form the same angle, not exceeding $\pi/2$, with the segment from x to y. If (x, dx) and (y, dy) do not satisfy this condition then there is a continuum of arcs consisting of two helicoidal arcs connecting (x, dx) and (y, dy). If the osculating planes at (x, dx) and (y, dy) are prescribed and the two helicoidal arcs are required to have at their joining point not only the same tangent but also the same osculating plane, then the solutions become discrete; in particular, there are exactly two with minimal total curvature. *H. Busemann* (Los Angeles, Calif.).

Argunov, B. I. Configurational postulates in projective planes and their algebraic equivalents. Vestnik Moskov. Univ. **1948**, no. 1, 47–51 (1948). (Russian)

In this paper, the summary of a dissertation at the University of Moscow, the author discusses various configuration theorems and their algebraic equivalents. In the natural ring of coordinates introduced by the reviewer [Trans. Amer. Math. Soc. **54**, 229–277 (1943); these Rev. **5**, 72] a line of a projective plane has an equation $y=x\cdot mob$. Addition and multiplication are defined from $a\cdot 1ob=a+b$ and $a\cdot bo0=ab$. The author introduces some further notation, defining symbols for the solution x of the following equations: $b-a$ for $a+x=b$, $b\sim a$ for $x+a=b$, a^{-1} for $ax=1$, a^{-1} for $xa=1$, $c\sim b:a$ for $a\cdot xob=c$, and $c-a\cdot b$ for $a\cdot box=c$. He includes a large diagram showing the relationships of various configuration theorems on both a projective and an affine basis. He also finds the "local" algebraic equivalents of the configuration theorems. Thus the Desargues theorem is locally equivalent to $(ca):[ca\sim(cb-c)]=a:[a\sim(b-1)]$. *M. Hall, Jr.* (Columbus, Ohio).

Lagrange, René. Sur les produits d'inversion. C. R. Acad. Sci. Paris **226**, 866–868 (1948).

En prolongement de sa note précédente [mêmes C. R. **226**, 625–627 (1948); ces Rev. **9**, 459] l'auteur donne l'expression symbolique des constantes qui définissent le résultat d'un produit d'inversions, et étudie le cas où ce produit se réduit à une transformation euclidienne, à une homothétie, ou à l'identité. *P. Belgodère* (Paris).

Thébault, Victor. Sur de nouveaux points de tétraèdre. C. R. Acad. Sci. Paris **227**, 754–755 (1948).

Krames, Josef. Über Parallaxeneigenschaften windschiefer Geraden. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 219–232 (1948).

The parallax of two skew lines with respect to a given direction is the distance between the given lines measured along a transversal having the given direction. The author introduces this definition and derives simple properties of the parallax. *E. Lukacs* (China Lake, Calif.).

Krames, Josef. Parallaxeneigenschaften zweier Sehstrahlbündel. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 233–246 (1948).

The author generalizes slightly a theorem obtained in an earlier paper. Statement 5 of that paper [Monatsh. Math. Phys. **49**, 327–354 (1941); these Rev. **3**, 300], originally derived for lines of parallax zero, is extended to lines of any parallax. For the definition of parallax see the preceding review. *E. Lukacs* (China Lake, Calif.).

Convex Domains, Extremal Problems

Hadwiger, H. Zum Problem des vollständigen Ungleichungssystems bei konvexen Körpern. Arch. Math. **1**, 13–17 (1948).

The system of inequalities (not yet completely known) between the volume V, the surface area F and the total mean curvature M of a convex body in ordinary space is obtained for two subclasses of the class of all convex bodies. These subclasses are defined as follows. Let $A(\lambda)$ denote the set of the inner and outer parallel bodies of a convex body A such that $A(0)$ is the kernel, i.e., the locus of the centers of the inscribed spheres, of A. G. Bol [Abh. Math. Sem. Hansischen Univ. **15**, 37–56 (1943); these Rev. **7**, 474] has proved that the right-hand derivatives $F'(\lambda)$ and $M'(\lambda)$ satisfy the inequalities $F'(\lambda)\geqq 2M(\lambda)$, $M'(\lambda)\geqq 4\pi$. Each of the two subclasses in question consists of the bodies for which equality is valid for all $\lambda\geqq 0$ in one of these inequalities. *W. Fenchel* (Copenhagen).

Straszewicz, S. Un théorème sur la largeur des ensembles convexes. Ann. Soc. Polon. Math. **21**, 90–93 (1948).

The thickness λ of a point set E in the Euclidean plane is the minimal distance between two parallel lines of support of E. The author proves the following theorem. If a convex set E with thickness λ is contained in the union of two arbitrary sets E_1 and E_2 with thicknesses λ_1 and λ_2, then $\lambda\leqq\lambda_1+\lambda_2$. *W. Fenchel* (Copenhagen).

Goldberg, Michael. Circular-arc rotors in regular polygons. Amer. Math. Monthly **55**, 393–402 (1948).

A rotor is a convex closed curve which remains tangent to all the sides of a fixed polygon during a complete rotation of the curve. The simplest examples are the curves of constant width, which rotate in any rhombus. It was proved by M. Fujiwara [Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **4**, 43–55 (1915)] that in every other case the fixed polygon must be regular. The author simplifies the discussion by giving a general construction for rotors composed of circular arcs. For the triangle or square these are Rouleaux's figures consisting of two or three equal arcs inclined at 60°. The pentagon needs four arcs with two different radii, and the hexagon seven arcs with three different radii.
H. S. M. Coxeter (New York, N. Y.).

Inzinger, Rudolf. Über konvexe ebene Bereiche, die eine einparametrige Schar von Grösstdreiecken besitzen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 263–285 (1948).

For a given bounded closed convex domain B with interior points in E^2 call minimal triangle, abbreviated as min. t. (or maximal triangle, abbreviated as max. t.) a triangle of minimal (maximal) area that contains (is contained in) B. If B is bounded by an ellipse then it possesses a one parameter family of min. t. and of max. t. Moreover, the max. t. of B are at the same time the min. t. of a concentric and similar ellipse. For any B the centers of the sides of a min. t. are the vertices of a max. t. and conversely. There are other convex curves than ellipses with a one parameter family of min. t. (and therefore also of max. t.). But if B has a one parameter family of min. t. and if these min. t. are at the same time max. t. of another domain B', then B and B' are bounded by concentric similar ellipses. *H. Busemann.*

Fejes Tóth, László. On ellipsoids circumscribed and inscribed to polyhedra. Acta Univ. Szeged. Sect. Sci. Math. **11**, 225–228 (1948).

The paper gives a new contribution to a problem treated formerly by the author [Bull. Amer. Math. Soc. **54**, 139–146 (1948); these Rev. **9**, 525]. Consider all polyhedra with a given number n of vertices (faces). Let E_n and e_n denote the volumes of the circumscribed and the inscribed ellipsoids of such a polyhedron, i.e., the ellipsoids containing (contained in) the polyhedron and having minimal (maximal) volume. The author investigates the minimum of E_n/e_n for prescribed n. In the present paper it is shown that the two

ellipsoids giving this minimum are concentric. The corresponding ellipses for polygons in the plane are always homothetic. By an example it is shown that this is not always the case with the extremal ellipsoids.

W. Fenchel (Copenhagen).

Tricomi, Francesco. Sul volume compreso tra due superficie parallele. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **81–82**, 197–204 (1948).

The author proves the formula of Steiner for the volume between a surface and one of its parallel surfaces in a slightly more general form, i.e., for the volume between two parallel surfaces, and remarks that if these parallel surfaces have opposite, numerically equal distances from the original surface, the mean curvature of the latter does not occur in the formula. *W. Fenchel* (Copenhagen).

Pólya, George. Torsional rigidity, principal frequency, electrostatic capacity and symmetrization. Quart. Appl. Math. **6**, 267–277 (1948).

Four theorems on Steiner's process of symmetrization in two and three dimensions are proved by elementary means. For two-dimensional figures, the author shows that symmetrization leaves the area unchanged and decreases the perimeter. In three dimensions, symmetrization leaves the volume unchanged and decreases the surface area. These results lead at once to certain inequalities on integrals related to a function $z=f(x, y)$ which is positive within a given domain D and vanishes along the boundary of D. These inequalities are used to prove the following physical statements: (1) symmetrization increases the torsional rigidity of a prismatic bar, (2) symmetrization decreases the principal frequency of a uniformly stretched membrane, (3) symmetrization decreases the electrostatic capacity of a condenser formed by two infinite cylinders, one contained within the other.

By using the known results for a rectangle, the author is able to give upper and lower bounds for the torsional rigidity of a rhombus. He concludes that of all quadrilaterals with a given area, the square has the maximum torsional rigidity and the minimum principal frequency. It is also shown that of all plane domains with a given area, the circle has the maximum torsional rigidity and the minimum principal frequency. The complete proof of the last statement is not given in this paper. *G. H. Handelman.*

Algebraic Geometry

Tognetti, Mario. Sul gruppo delle cuspidi delle curve cuspidate di una rete. Pont. Acad. Sci. Acta **6**, 237–239 (1942).

Étant donné sur une surface algébrique F un système linéaire à trois paramètres de courbes $|C|$, parmi les courbes du système, certaines ont un rebroussement R. Le lieu des points R est une courbe H; tout réseau extrait de $|C|$ donne lieu sur H à un groupe G de points R. En supposant $|C|$ irréductible, simple, dépourvu de courbes fondamentales et de points-bases, l'auteur démontre la relation d'équivalence fonctionnelle: $Q=M(H)+12(C, C)-52(C, C')-16\Omega$ où $M(H)$ est un groupe canonique effectif de H, Ω une série invariante de F, (C, C) et (C, C') les séries covariantes de F. [Ces notions ont été introduites par F. Severi, voir l'analyse suivante.] *L. Gauthier* (Nancy).

★Severi, Francesco. Serie, Sistemi d'Equivalenza e Corrispondenze Algebriche sulle Varietà Algebriche. Vol. 1. Edizioni Cremonese, Rome, 1942. vi+415 pp.

The theory of series and systems of equivalence on algebraic varieties was initiated by Severi, and this first volume of a complete account of the subject is very welcome. It is beautifully printed, fully documented and a pleasure to read. The author allows himself to wander down many pleasant bye-paths, so that an almost complete survey of the algebraic geometry of curves and surfaces can be obtained by the diligent reader. The ample content can best be indicated by the chapter headings: I, Transformations, varieties, intersections of varieties. II, Digression on the invariantive order of varieties. III, Fundamental concepts on series and systems of equivalence. IV, Rational series and systems on a surface; the fundamentals of algebraic correspondences between varieties. V, Series of equivalence on reducible curves. VI, Invariantive series of equivalence on a surface. VII, The Riemann-Roch theorem on a surface.

D. Pedoe (London).

Severi, Francesco. Ulteriori sviluppi della teoria delle serie di equivalenza sulle superficie algebriche. Pont. Acad. Sci. Comment. **6**, 977–1029 (1942).

Series of sets of points on an algebraic surface were first introduced and discussed by Severi in 1932 and have formed the subject matter of many papers since [cf. the preceding review]. This paper gives full references and completes the author's researches on this subject. *D. Pedoe* (London).

★Severi, Francesco. Fondamenti di Geometria Algebrica. CEDAM, Padova, 1948. iii+172 pp.

This is an important work for anyone who wishes to obtain a rapid survey of Italian algebraic geometry, especially the theory of algebraic surfaces. It is well written, and although proofs are usually omitted the sequence of ideas is always clear.

The book would have been greatly improved if detailed references were given, not only to the author's own work, which is both considerable and important, but to that of other geometers. Based on a course of lectures to students at the Scuola Normale Superiore di Pisa, the book frankly sets out to remind Italians of the great contributions made to the study of geometry by Italian geometers. Nobody would wish to deprecate this aim, but it seems fairly clear that progress can now best be made by uniting the Italian discipline with other disciplines. The author goes part of the way; for instance Lefschetz, van der Waerden, Hodge and Todd are mentioned, but detailed references are only given when the author can claim that enough work has been done on certain theorems in Italy to make their non-Italian origin comparatively unimportant. Thus, since a section of the book deals with the author's improvements on the work of Lefschetz, a proper reference is given to the latter's work; the work of van der Waerden has been equaled by the author, so a reference (Math. Annalen—1927, 1928, 1929) is given. Todd is mentioned in a footnote, without reference, and since Italian priority cannot be claimed for Hodge's theorem on the topological interpretation of the geometric genus p_g of a surface we are merely told that the theorem is due to Hodge, no date or reference being given. On another page, where the author claims to have enunciated a theorem on double integrals of the first kind on an algebraic surface which Hodge proved subsequently the reference is Hodge (1930). Surprisingly enough a reference to the author's prior work is omitted in this case.

The scope of the book is indicated by the chapter headings. I, Algebraic varieties—intersections; II, Linear systems on an M_k; III, Geometry on a surface; IV, Invariant and covariant linear systems on a surface; V, Invariant and covariant series of equivalence on a surface; VI, Theorem of Riemann-Roch on a surface; VII, Continuous systems of curves; VIII, The topological and transcendental aspects of geometry on a surface. *D. Pedoe* (London).

Severi, Francesco. Le varietà multiple diramate e il loro teorema di esistenza. Memorias de Matemática del Instituto "Jorge Juan," no. **4**, 17 pp. (1946).

The main object of this paper is the proof of an existence theorem for algebraic functions of r variables, represented multiply on an algebraic variety with given branching. Detailed references to the work of Enriques, Zariski and van Kampen are given. [For an easily accessible report on the background see Zariski, Algebraic Surfaces, Ergebnisse der Math., v. 3, no. 5, Springer, Berlin, 1935, chap. 8]. Among the theorems proved are: (1) two varieties V, $\bar{V}$, represented n-ply on a variety M with the same branch variety and isomorphic monodromy groups are birationally equivalent; (2) the members of an irreducible algebraic system of varieties represented n-ply on a fixed variety with fixed branching are birationally equivalent.

In the final section there is a discussion of the Poincaré fundamental group attached to S_r-D, where S_r is a linear space $S_r(x_1, x_2, \cdots, x_r)$, and D a primal in the space. A system of elementary relations (*) of the group is obtained and the following existence theorem proved. Necessary and sufficient conditions for the existence of an irreducible function with n branches of the variables $x_1, x_2, \cdots, x_r$, having a given branch variety D in $S_r(x_1, x_2, \cdots, x_r)$, and given substitutions on the generating cycles of the Poincaré group G attached to S_r-D, are that these substitutions satisfy (*) and generate a transitive group. *D. Pedoe* (London).

Severi, Francesco. Sul gruppo di monodromia d'uno spazio lineare multiplo diramato. Boll. Un. Mat. Ital. (3) **3**, 1–3 (1948).

An irreducible algebraic variety V_k^n, of order n and dimension k, lying in S_r, is projected from a generic S_{r-k-1} on to a generic S_k, so that the variety is represented n-ply on S_k, with a branch variety D. It is proved that the monodromy group of this projection is the complete symmetric group of degree n. *D. Pedoe* (London).

Morton, V. C., and Chapple, M. T. A point representation of a system of space cubic curves which pass through four given points and whose chords belong to a given tetrahedral complex. Quart. J. Math., Oxford Ser. **19**, 133–139 (1948).

By the ∞^1-∞^1 correspondence between points $T(X_T)$, $t(x_t)$ in threefold space given by the transformation

$$X_T=ax_t/(a+\theta), \text{ etc.}; \quad x_t=(a+\theta)X_T/a, \text{ etc.}$$

(where θ is variable), a point T corresponds to a line τ of a tetrahedral complex K whose singular points are the vertices 1234 of the tetrahedron of reference, and a point t to a cubic curve ρ passing through 1234 and whose chords belong to K. The following correspondences are obtained between loci of points t and families of curves ρ. General plane: Curves ρ with a fixed chord. General line: Curves ρ lying on a fixed quadric through 1234. Scroll belonging to K: Curves ρ which meet the curve C_p^n, locus of points T corresponding to generators of the scroll. Focal surface of congruence belonging to K: Curves ρ touching the surface G, locus of points T corresponding to lines of the congruence.

The characters of the scroll and congruence corresponding to a general curve C_p^n and a general surface G of order n are investigated, and the following numbers obtained of curves ρ which: touch C_p^n: $4(n+p-1)$; meet C_p^n three times: $\frac{2}{3}(n-1)(n-2)(2n-3)-2(n-2)p$; have four point contact with G: $4n(n-1)(3n-2)$; touch G at one point and osculate it at another: $2n(n-1)(n^3+8n^2-20n+12)$; touch G at three points: $\frac{4}{3}n(n-1)(n-2)(n^3+3n^2-8n+6)$.
P. Du Val (Istanbul).

Tanturri, Giuseppe. Su alcuni inviluppi di rette. Boll. Un. Mat. Ital. (3) **3**, 46–48 (1948).

Étant donnée une quartique plane générique C, l'enveloppe des droites qui la coupent suivant un quaterne de rapport anharmonique donné ρ, est une enveloppe de douzième classe non décomposée (qui se réduit à C pour $\rho=0$). L'auteur détermine ici les quartiques non décomposées C, telles que les enveloppes obtenues soient décomposées quelque soit ρ: si C admet un point triple, toutes les enveloppes le contiennent comme partie fixe (avec un ordre de multiplicité qui dépend de sa structure); si C admet trois points de rebroussement, les composantes des enveloppes sont les enveloppes elliptiques de troisième classe ayant les mêmes rebroussements que C (avec mêmes tangents); il n'y a pas d'autre cas. *L. Gauthier* (Nancy).

Segre, B. Intorno agli S_k che appartengono alle forme generali di dato ordine. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 261–265 (1948).

A necessary and sufficient condition that the general hypersurface F^n of S_r carry straight lines is that $n\leqq 2r-3$, a condition which may be easily surmized, but which was first rigorously established by van der Waerden [Math. Ann. **108**, 253–259 (1933)]. Pursuing the similar but more difficult question for S_k's, $k>1$, the author finds that a necessary and sufficient condition that the general F^n in S_r carry some S_k ($n\geqq 1$, $k\geqq 0$) is that $r\geqq k+(k+1)^{-1}\binom{n+k}{n}$, except in the case that $n=2$; $k\geqq 2$, in which case the condition is $r\geqq 2k+1$. The necessity is proved by setting up the correspondence between F^n and S_k in which F^n and S_k correspond if S_k lies on F^n, and applying the principle of counting constants. A simple induction on k shows the sufficiency of the condition $r\geqq k+\binom{n+k-1}{n-1}$. The deduction of the previously stated condition is left for a future note. *A. Seidenberg.*

Longo, C. Sui sistemi di ipersuperficie di S_r che ammettano lo stesso sistema primo polare nei casi in cui l'omografia determinata dai poli sia particolare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 536–541 (1947).

L'auteur poursuit et achève l'étude entreprise par lui [mêmes Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 282–287 (1947); ces Rev. **9**, 611], de tous les systèmes d'hypersurfaces de S_r qui ont le même système de premières polaires. Lorsque l'homographie Ω entre les pôles est singulière et admet une seule racine caractéristique, l'auteur donne une méthode récurrente de détermination de l'équation de l'hypersurface V_{r-1}^n cherchée. Lorsque l'homographie Ω est singulière et admet plusieurs racines caractéristiques, on obtient le système linéaire des hypersurfaces solutions en sommant les solutions qui correspondent aux différentes racines, prises dans les sous-espaces qui leurs sont relatifs. Ce dernier point, annoncé par l'auteur, sera démontré dans un mémoire ultérieur. *L. Gauthier* (Nancy).

Godeaux, Lucien. Sur les surfaces semi-canoniques de l'espace ordinaire. Univ. Nac. Tucumán. Revista A. **6**, 81–89 (1947).

L'auteur appelle surface semi-canonique une surface sur laquelle le système canonique complet est découpé par les quadriques, et donne une méthode pour determiner une telle surface. En particulier il montre que la surface d'ordre dix, ayant une courbe double d'ordre vingt et genre 15, possédant seize points triples pour la courbe et pour la surface, est une surface semi-canonique, la courbe double étant supposée tracée sur une surface du quatrième ordre.

M. Piazzolla Beloch (Ferrara).

Differential Geometry

***Haack, Wolfgang. Differential-Geometrie. Teil II.** Wolfenbütteler Verlagsanstalt G.m.b.H., Wolfenbüttel and Hannover, 1948. 131 pp.

This volume falls into three parts. The first part, pages 7–22, is an introduction to the Ricci calculus. The usual topics are considered in connection with the surface theory studied in volume 1 [cf. these Rev. **9**, 612]. For instance, there are sections on the Christoffel symbols, the covariant derivatives of tensors, the Riemann curvature tensor, and on the integrability relations of surface theory.

The second part, pages 33–50, contains an introduction to the invariant derivatives of surface theory, and, among other things, as an illustration of the application of this theory, a study of the central surfaces associated with a given surface.

The remainder of the book, pages 51–128, is devoted to differential line geometry. In an introductory chapter, Plücker's line coordinates, Study's dual vectors (used everywhere thereafter), reguli, linear congruences, and linear complexes are presented. In a chapter on ruled surfaces, the moving trihedral of the ruled surface is set up and the surfaces determined by the vectors of the trihedral are studied. In a chapter on two-dimensional manifolds of lines, the author discusses such topics as the quadratic fundamental forms, focal planes and focal points, the Gauss-Bonnet integral formula, isotropic manifolds, and Laplace sequences, using the language of the Ricci calculus. In a last chapter, on three-dimensional manifolds, such topics as the fundamental forms, the first order neighborhood of a line, the complex of principal normals, and the characteristic ruled surfaces are studied, this time in the language of invariant derivatives.

The goal of this volume is to prepare the reader for further study rather than to describe the subjects studied in complete detail. Like the first volume, it is clearly written and well motivated throughout. There are references to more complete texts and there is an index, but there are no problems for the reader. *A. Schwartz* (New York, N. Y.).

Marcantoni, A. Sopra alcune formule relative alla rappresentazione gaussiana della sfera sul piano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 70–75 (1948).

The author reviews the conformal Gaussian projection of an ellipsoid on a plane corresponding to a transverse Mercator projection with poles on the equation. The several cartographic aspects of the projection, such as distortion, arc length and azimuth correlation, are outlined.

N. A. Hall (Minneapolis, Minn.).

Segre, B. Una nuova caratterizzazione della sfera. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 420–422 (1947).

A surface S in the ordinary space is a plane or a sphere if and only if $\oint(\tau/\kappa)ds=0$ for every closed curve on S, where s, κ, τ denote the arc length, the curvature, the torsion of the curve, respectively. This is analogous to a theorem of Scherrer [Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolph Fueter), 40–46 (1940); these Rev. **3**, 89]. *W. Fenchel* (Copenhagen).

Segre, B. Sulla torsione integrale delle curve chiuse sghembe. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 422–426 (1947).

Let C denote a skew curve (open or closed) in ordinary space. The author defines a measure for the deviation of C from being plane as twice the spherical diameter of the minimal spherical cap containing the binormal indicatrix D of C, if there is a half sphere containing D, and as 2π, if this is not the case. This number is denoted by a and called the twisting (torcimento) of C. If α denotes the length of an arc of a great circle with the property that every great circle passing through a point of this arc intersects D, then $a\geqq 2\alpha$. Let now C be closed, s its arc length and τ its torsion. Then it is proved that the total torsion $\oint|\tau|ds$, i.e., the length of D, is greater than a. This follows easily from a theorem due to the author [Boll. Un. Mat. Ital. (1) **13**, 279–283 (1934)] stating that a closed curve of length $l\leqq 2\pi$ on the unit sphere is contained in a cap of spherical diameter $l/2$. An example with $\alpha=\pi/2$ shows that the inequality $\oint|\tau|ds>a$ is the best possible for curves with $a\geqq\pi$.

W. Fenchel (Copenhagen).

Gachet, Henry. Sur les surfaces et les courbes déformables. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **23**, 145–154 (1948).

The author calls attention to the notion of "normal velocity" of a moving surface S or a moving curve C. Let P be a point of S (C). The velocity vector of every moving point of S (C) which passes P at the moment t has at P the same component normal to S (C). This component is called the normal velocity of S (C) at P. As an application the following problem is considered. At each instant t a point leaves a fixed point O with a constant velocity in a variable direction in a vertical plane and moves under gravity. Then the trajectory is a moving curve T. A construction of the normal velocity of T is given. By means of this normal velocity the author constructs the tangent of γ, γ being the curve formed by the particles at a certain time τ.

J. Haantjes (Leiden).

Graf, H., und Thomas, H. Zur Frage des Gleichgewichts von Vierecksnetzen aus verknoteten und gespannten Fäden. II. Rückungsfadennetze mit isotroper Spannungsverteilung und rhombischer Netzstruktur auf den Scherkschen Minimalflächen und auf den Wendelschraubenflächen. Math. Z. **51**, 166–196 (1948).

In a preceding paper [Math. Z. **48**, 193–211 (1942); these Rev. **6**, 20], the authors studied certain nets on surfaces [surface: topological image of a plane convex domain; net: topological image of the parallels to the axes in the domain]. We imagine the lines of the net to be realized by one-dimensional unstretchable threads that are knotted together at their intersections. These threads are assumed to bend without resistance and to slide on the surface without fric-

tion. Tangential forces at their endpoints keep the net stretched tight in the shape of the surface. A G-net satisfies an additional condition: at each point of the surface, the two threads through that point have the same arc element and the same tension. The authors had discussed four types of G-nets, the last one being conjugate G-nets with constant arc elements. Such T-nets necessarily are translation nets on minimal surfaces. The only minimal surfaces with [real] translation nets are the Scherkian minimal surfaces $\tanh z = \tan (x \csc \gamma) \cdot \tan (y \csc \gamma)$, $0 < \gamma < \pi/2$, and the helicoid $z = y \tan cx$.

In the present paper, the converse statement is proved: the translation nets of these minimal surfaces can always be interpreted as T-nets. To this end, these surfaces and their ∞^1 translation nets are investigated in detail. The helicoid is dealt with as a simply periodic limit case of the double periodic Scherkian surfaces. Each of the nonplanar translation nets of the latter is associated to one of its asymptotic lines: each of its curves is the locus of the midpoints of the secants of that asymptotic line through a fixed point on that line. The equations of the nets are given in terms of Jacobian elliptic functions.

In a final chapter, finite models of the four types of G-nets are discussed. *P. Scherk* (Saskatoon, Sask.).

Niče, V. Les éléments imaginaires en géométrie. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **1**, 193–208 (1946). (Croatian. French summary)

L'auteur fait connaître certains résultats, déjà publiés ou à paraître, relatifs à certaines courbes ou certaines surfaces réglées ou non. Ces résultats sont obtenus par l'étude de la figure envisagée en relation avec les différents éléments imaginaires de première espèce (points, droites, plans, cônes, coniques et en particulier conique absolue), ces éléments étant, le plus souvent, introduits d'après Staudt à l'aide d'involutions. Après quelques indications relatives aux tangentes imaginaires des courbes gauches ou aux couples de génératrices imaginaires des surfaces réglées, l'auteur envisage plus spécialement certaines surfaces d'ordre trois ou quatre contenant la conique absolue, leurs sections planes et les foyers quadruples de ces dernières. Il introduit les ombilics d'une surface réglée comme les intersections (lorsqu'elles existent) des couples de génératrices isotropes de la surface et discute leur existence pour les surfaces réglées du troisième ordre avec examen spécial des particularités auxquelles donne lieu le conoïde Plücker. Il montre que pour les surfaces réglées d'ordre 4 le nombre des ombilics ne peut dépasser 4, et que pour certaines de ces surfaces il existe une relation intéressante entre les ombilics et les sections planes circulaires. Suit le dénombrement des surfaces réglées d'ordre trois ou quatre appartenant à une congruence linéaire et admettant des ombilics, puis une étude des courbes circulaires d'ordre 3 ou 4 et de genre 0 en relation avec la génération que l'on peut donner de ces courbes au moyen d'une inversion quadratique plane généralisée. La considération de l'inversion généralisée de l'espace conduit enfin à une étude géométrique assez poussée des surfaces d'ordre 3 à point double et des surfaces d'ordre 4 avec point et cercle doubles. *P. Vincensini* (Besançon).

Blazina, Jakov. Bertrand'sche Raumkurven. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **3**, 11–16 (1948). (Croatian. German summary)

Exposition of theorems due to Bertrand, Serret, Schell and Mannheim.

Santaló, L. A. *D* curves on cones. Math. Notae **7**, 179–190 (1947). (Spanish)

On sait qu'on appelle courbes D (de Darboux) d'une surface les courbes dont la sphère osculatrice est en chaque point tangente à la surface. L'auteur recherche et étudie les courbes D des cônes. Il commence par donner, sans forme vectorielle, l'équation différentielle (du deuxième ordre) des courbes D d'une surface quelconque, et montre que pour les cônes cette équation s'intègre par quadratures. La façon dont les constantes d'intègration figurent dans le résultat met en évidence le fait que, si l'on considère la trace du cône sur une sphère de rayon un centrée au sommet (courbe base du cône), une courbe D quelconque coupe sous le même angle toutes les génératrices du cône correspondant à une même valeur de la courbure de la courbe base. Il résulte de là que la condition nécessaire et suffisante pour qu'un cône admette pour courbe D une loxodromie est qu'il soit de révolution (toutes ses courbes D sont alors des loxodromies). Les courbes D d'un cône peuvent affecter des formes différentes, l'auteur discute ces formes en distinguant les deux cas où D coupe ou non toutes les génératrices du cône. *P. Vincensini* (Besançon).

Bompiani, E. Fasci di elementi differenziali nel piano proiettivo. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 124–168 (1948).

This is a contribution to the study of point transformations between two projective planes in the neighborhood of corresponding points O, O' with a view toward approximating them by Cremona transformations. If the latter is possible up to a certain order k in the neighborhoods of O, O' then two bundles of rational curves through O, O', respectively, are determined whose differential elements E_k of order k fix the point transformation up to order k. Thus the author is led to study systems of ∞^1 E_k's through a point O which can belong to a bundle of rational curves. His first problem is to characterize them geometrically. The second is: given an admissible bundle of E_k's, to determine the minimum order of a bundle of rational curves containing them. These questions are answered in detail for $k=3$ ($k=2$, already known, is also given for completeness). The discussion divided into cases according to the number of distinct inflectional tangents and to whether or not these directions are hyperinflectional. For any admissible bundle of E_3's the bundle or bundles of rational curves of minimum order are distinguished not only by their order which is always less than or equal to four but also by certain projective invariants. The paper thus demonstrates the possibility of approximating a point transformation in the neighborhood of third order of two corresponding points by a Cremona transformation whose homaloidal system consists at most of quartics. *J. L. Vanderslice.*

Bompiani, E. Isometria di calotte superficiali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 274–294 (1948).

This is an investigation of the isometry of calottes (surface "patches" of order k) and its connection with the classical theory of applicability of surfaces. The major conclusions are as follows. (1) Any two second-order calottes are applicable. (2) Necessary and sufficient for applicability of two third-order calottes is the equality of Gaussian curvature of the second-order calottes belonging to them, the applicability being possible in infinitely many ways. (3) Necessary and sufficient for applicability of two fourth-

order calottes is equality of curvature and its first derivatives. Also included is the actual geometric and analytic construction of the isometries guaranteed by result (2).

J. L. Vanderslice (College Park, Md.).

Bompiani, E. Monoidi del 3° ordine per una calotta superficiale del 4° ordine. Monatsh. Math. **52**, 190–193 (1948).

Dans des notes antérieures, où il a étudié la distribution des éléments curvilignes du quatrième ordre issus d'un point d'une surface (formant la calotte σ_4 de la surface ayant le point pour centre), l'auteur a caractérisé géométriquement le repère invariant intrinsèquement lié à la surface pour lequel σ_4 admet la représentation $z=xy+\frac{1}{3}(x^3+y^3)+\frac{1}{3}(x^3+y^3)(\alpha x+\beta y)+\cdots$. La considération des monoïdes du troisième ordre contenant σ_4 lui permet, comme il est montré dans la note actuelle, d'obtenir une nouvelle caracterisation de ce même repère en même temps qu'un certain nombre de résultats géométriques nouveaux. Partant de la représentation ci-dessus de σ_4, l'auteur considère les surfaces du troisième ordre F^3 passant par la calotte et ayant un point double (monoïdes). Il montre que par tout point de l'espace, non situé dans le plan $z=0$ tangent à σ_4 en son centre, il passe une F^3 et une seule ayant le point considéré pour point double. Les monoïdes F^3 passant par σ_4 pour lesquels le point double est biplanaire sont en nombre ∞^2, le lieu du point double étant une surface du quatrième ordre admettant le centre de σ_4 pour point uniplanaire et tangente à σ_4 en ce même point. Pour ceux des monoïdes précédents pour lesquels l'intersection des plans tangents au point double coupe la polaire de Darboux de la droite joignant ce point au centre de σ_4, le lieu des points biplanaires des monoïdes considérés est une sextique rationnelle. Cette sextique a le centre de σ_4 pour point triple; d'autre part elle est située sur un cône cubique admettant la droite $x=\beta z$, $y=\alpha z$ comme droite double, et cette remarque met en évidence le caractère invariant des coefficients α et β. Si l'on ajoute la condition supplémentaire que l'intersection de l'un des plans tangents au point biplanaire avec le plan tangent $Z=0$ à σ_4 soit la polaire de Darboux dont il est question plus haut, ou trouve que le point double biplanaire est déterminé, et que c'est le sommet du tétraèdre de référence opposé au plan $Z=0$. Ce tétraèdre reçoit ainsi une caractérisation géométrique qui est la nouvelle caractérisation annoncée. L'équation de la trace du deuxième plan tangent au point biplanaire sur $Z=0$ est $\alpha X+\beta Y-T=0$. Cette équation met à nouveau en évidence le caractère invariant de α et β, et donne une signification géométrique à l'annulation des invariants α et β, cette annulation exprimant que le point double du monoïde F^3 est uniplanaire. La recherche générale des F^3 à point double uniplanaire passant par σ_4, par laquelle se termine la note, conduit à ce résultat qu'il existe quatre points de l'espace uniplanaires pour quatre F^3 contenant σ_4, et que, si l'on choisit l'un de ces points comme sommet du tétraèdre de référence opposé a $Z=0$ et le plan tangent double au point uniplanaire pour $T=0$, on obtient pour la surface un développement canonique sans termes du quatrième ordre.

P. Vincensini (Besançon).

Greer, E., and Bell, P. O. A study of analytic surfaces by means of a projective theory of envelopes. Trans. Amer. Math. Soc. **64**, 253–267 (1948).

Étant donnée une famille de quadriques associées à chaque point d'une surface et y ayant un contact du deuxième ordre, soit $\mathfrak{C}_\lambda$ le cône ayant comme sommet un point x de la surface et comme directrice la caractéristique de la quadrique associée à x quand ce point subit un déplacement infinitésimal le long d'une courbe C_λ. Les auteurs trouvent les propriétés caractéristiques de certaines familles de quadriques et de courbes C en relation avec certaines configurations liées à la surface, telles que le faisceau canonique, certaines familles d'hypergéodésiques, la transformation de Čech, les pangéodésiques, la directrice de Wilczynski, etc. Par exemple, on trouve des caractérisations géométriques pour les quadriques de Moutard, celles de Davis et celles de Bompiani. Toutes ces familles de quadriques apparaissent comme caractérisées par la propriété commune de leur cônes caractéristiques d'être tangents au plan tangent à la surface le long d'une droite qui fait avec les asymptotiques et la direction de la tangente à C un birapport constant égal à $+1$ pour les quadriques de Moutard, à -1 pour celles de Davis, à 0 et ∞ pour celles de Bompiani.

M. Haimovici (Jassy).

Farina, Laura. Contributo allo studio locale delle trasformazioni puntuali fra due piani. Pont. Acad. Sci. Acta. **8**, 19–28 (1944).

It is well known that in a point-to-point transformation between two projective planes there is, at each pair of corresponding points, a triple of corresponding inflexional directions. The author studies the cases in which two or three of these inflexional directions coincide, giving the canonical developments of a transformation in the neighborhood of such a pair of points and the geometrical meaning of the invariants.

E. Bompiani (Pittsburgh, Pa.).

Villa, M. Le trasformazioni puntuali fra due spazi lineari. I. Intorno del 2° ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 55–61 (1948).

Villa, M. Le trasformazioni puntuali fra due spazi lineari. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 192–196 (1948).

Villa, M. Le trasformazioni puntuali fra due spazi lineari. III. Trasformazioni cremoniane osculatrici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 295–303 (1948).

These papers deal with a correspondence between two projective spaces S_r, S_r' in the neighborhood of two regular corresponding points O, O'. If the correspondence is not osculated by a projectivity, there are 2^r-1 inflexional elements of the second order (E_2) to which correspond also inflexional elements: the correspondence determines on each inflexional line a characteristic projectivity. Some of these results were already known.

If $r=2$, to an inflexional element of the third order (four infinitely near points on a line) through O' corresponds an element E_3 through O and it is known [Bompiani] that this E_3 determines a projectivity between the points of the tangent to E_3 and the pencil of lines through O. Therefore on each tangent there are three points corresponding to the inflexional lines through O. As a consequence it is possible to determine intrinsically a system of reference in the two planes (connected with the neighborhoods of the third order of O and O').

If $r>2$, let us consider the S_{r-1}' connecting $r-1$ inflexional directions $s_1', \cdots, s_{r-1}'$ through O': let V_{r-1} be the manifold corresponding to S_{r-1}' in S_r, and consider the intersection of the plane of two inflexional directions p_1, p_2 through O (different from $s_1, \cdots, s_{r-1}$) with V_{r-1}. The element E_3 of

this intersection determines (as above) a projectivity between the point-range on its tangent and the pencil of lines through O in the plane (p_1p_2) and therefore two points, corresponding to p_1 and p_2, on the tangent. With the help of these points it is possible to determine intrinsically a system of reference, and therefore canonical equations for the correspondence (depending on the neighborhood of the third order).

It is possible to osculate the given correspondence with $\infty^{r^2(r-1)/2}$ Cremona transformations of a particular type (product of r reciprocities); and also to determine one of these transformations with further conditions.

E. Bompiani (Pittsburgh, Pa.).

Vaona, Guido. Elementi differenziali d'iperosculazione di due trasformazioni puntuali. Boll. Un. Mat. Ital. (3) 3, 40–46 (1948).

Given two point-transformations T_1, T_2 between two projective spaces of dimensionality r, such that T_1, T_2 coincide up to the neighborhoods of order s (included) of two corresponding points O, O', there are $\{(s+1)^r+1\}/s$ differential elements E_k of order k through O (or O') such that every E_{s+k} through E_k is transformed by T_1 and T_2 into the same E_{s+k}. *E. Bompiani* (Pittsburgh, Pa.).

Kodaira, K. Relations between harmonic fields in Riemannian manifolds. Math. Japonicae 1, 6–23 (1948).

The object of this paper is to generalise the idea of a rational function on a Riemann surface by considering a class of integrals on a Riemannian manifold M of dimension n, which are harmonic everywhere except at a compact set S which is nowhere dense on M. A number of definitions and existence theorems are first required. (1) On the basis of certain existence theorems not proved in this paper a ρ-form $e_{(\lambda_1,\cdots,\lambda_\rho)}=(1/\rho!)e_{\lambda_1\cdots\lambda_\rho|i_1\cdots i_\rho}dx^{i_1}\cdots dx^{i_\rho}$ is associated with any point ξ of M and any set of integers $\lambda_1,\cdots,\lambda_\rho$ $(1\leqq\lambda_i\leqq n)$, this form being harmonic at all points of M except $x=\xi$. When $n=2$, $\rho=1$, and the metric is of the form $\sigma(dx^2+dy^2)$, $\int e_{(1)}$ and $\int e_{(2)}$ behave like the real and imaginary parts of $[(x-\xi)+i(y-\xi)]^{-1}$, respectively; and when the metric is Euclidean $e_{(\lambda_1\cdots\lambda_\rho)}$ behaves near $x=\xi$ like the coefficient of $d\xi^{\lambda_1}\cdots d\xi^{\lambda_\rho}$ in

$$d_\xi d_x[r^{-n+2}\sum d\xi^{i_1}\cdots d\xi^{i_{\rho-1}}dx^{i_1}\cdots dx^{i_{\rho-1}}]$$

to within a constant multiple. (2) If C is any ρ-chain on M a ρ-form $e[C]$ is associated with it by the formula

$$e[C]=(1/\rho!)\int_C e_{(\lambda_1,\cdots,\lambda_\rho)}d\xi^{\lambda_1}\cdots d\xi^{\lambda_\rho};$$

$e[C]$ is harmonic everywhere except on the boundary of C. (3) A "pole" γ_m^ρ of order m is defined by the association of a point ξ of M and a tensor $\sigma^{(\lambda_1\cdots\lambda_\rho)i_1\cdots i_{m-1}}$. Then if $\phi=(1/\rho!)\phi_{i_1\cdots i_\rho}dx^{i_1}\cdots dx^{i_\rho}$ is a ρ-form, we define

$$\phi(\gamma_m^\rho)=\frac{1}{\rho!(m-1)!}\sigma^{(\lambda_1\cdots\lambda_\rho)i_1\cdots i_{m-1}}\phi_{\lambda_1\cdots\lambda_\rho,\,i_1,\,i_2,\,\cdots,\,i_{m-1}},$$

where $,_j$ denotes covariant differentiation at ξ. (4) Let $\gamma_{m_i}^\rho$ $(i=1,\cdots,a)$, $\epsilon_{n_i}^\rho$ $(i=1,\cdots,b)$ be two sets of poles, and let C_i $(i=1,\cdots,k)$, D_i $(i=1,\cdots,l)$ be two sets of ρ-chains. Then a ρ-form ϕ is said to be a multiple of the divisor $\mathbf{d}=\gamma_{m_1}^\rho\cdots\gamma_{m_a}^\rho C_1\cdots C_k/\epsilon_{n_1}^\rho\cdots\epsilon_{n_b}^\rho D_1\cdots D_l$ if (a) ϕ differs from a form which is everywhere harmonic on M by a linear combination (with constant coefficients) of $e(\epsilon_{n_i}^\rho)$ and $e[D_j]$; (b) $\phi(\gamma_m^\rho)=0$, $\int_{C_j}\phi=0$.

The main result of the paper is the following generalisation of the Riemann-Roch theorem for rational functions on a Riemann surface. If $\mathbf{d}$ is a divisor of the form $\gamma_{m_1}^\rho\cdots\gamma_{m_a}^\rho C_1\cdots C_k/\epsilon_{n_1}^\rho\cdots\epsilon_{n_b}^\rho$, and A is the number of ρ-forms ϕ which are multiples of $\mathbf{d}$ and are such that both ϕ and ϕ^* have all their periods zero, then $A=b-a-k-R_\rho+B$, where R_ρ is the ρth Betti number of M, and B is the number of linearly independent ρ-forms which are multiples of $\mathbf{d}^{-1}$. In the case in which $n=2\nu$ is even, the theorem is extended to the case of ν-forms which satisfy in addition the condition $\phi=\pm(i)^\nu\phi^*$, where ϕ^* is the dual of ϕ. These forms are called Abelian forms, on account of their analogy with Abelian forms on a Riemann surface. *W. V. D. Hodge.*

Chern, Shiing-shen. Note on projective differential line geometry. Acad. Sinica Science Record 2, 137–139 (1948).

This note contains some elementary notions on the existence of ruled surfaces in real projective 3-space. According to a letter of Wen-tsun Wu to the author, such a surface, without self-intersections, is orientable. *H. Whitney.*

Chern, Shiing-Shen. Correction to my paper "Note on affinely connected manifolds." Bull. Amer. Math. Soc. 54, 985–986 (1948).

The paper appeared in the same Bull. 53, 820–823 (1947); these Rev. 9, 67.

Yen, Chih-Ta. Sur une connexion projective normale associée à un système de variétés à k dimensions. C. R. Acad. Sci. Paris 227, 461–462 (1948).

Dans cette note l'auteur annonce le théorème suivant. On peut définir de manière intrinsèque dans l'espace X_n une connexion projective associée à un système de variétés à k dimensions dans le cas où le système est formé par les variétés intégrales d'un système complètement intégrable d'équations aux dérivées partielles du deuxième ordre tel qu'il existe une variété intégrale et une seule tangente à un élément de contact arbitraire à k dimensions.

J. Haantjes (Leiden).

Egorov, I. P. On collineations in spaces with projective connection. Doklady Akad. Nauk SSSR (N.S.) 61, 605–608 (1948). (Russian)

Let x^ν be the coordinates of a point in the space X_n with symmetrical connection $\Gamma_{\beta\gamma}^\alpha=\Gamma_{\gamma\beta}^\alpha$ and let

$$\Pi_{\beta\gamma}^\alpha=\Gamma_{\beta\gamma}^\alpha-\frac{2}{n+1}\delta_{(\beta}^\alpha\Gamma_{\gamma)\sigma}^\sigma$$

$(\alpha,\beta,\gamma,\nu,\sigma=1,2,\cdots,n)$ be the projective connection of a path. The infinitesimal transformation (1) $XF=p_\sigma v^\sigma$, where $p_\sigma=\partial F(x^1,\cdots,x^n)/\partial x^\sigma$, $v^\sigma=v^\sigma(x^1,\cdots,x^n)$, defines a collineation which transforms the path into a path of the space X_n if v^σ is the solution of the system of equations

$$D\Pi_{\beta\gamma}^\alpha=v_{|\beta|\gamma}^\alpha-v^\sigma B_{\beta\gamma\sigma}-\frac{2}{n+1}v_{|\sigma|(\beta}^\sigma\delta_{\gamma)}^\alpha=0,$$

where

$$v_{|\beta}^\alpha=\partial_\beta v^\alpha-\Pi_{\beta\gamma}^\alpha v^\gamma,$$
$$\tfrac{1}{2}B_{\beta\gamma\delta}^\alpha=\partial_{[\gamma}\Pi_{\delta]\beta}^\alpha+\Pi_{\beta[\delta}^\sigma\Pi_{\gamma]\sigma}^\alpha,\qquad \delta=1,2,\cdots,n.$$

The set of r linearly independent operators (1) each of which defines a collineation in the space X_n with projective connection and which forms Lie's group of maximal order, is called the complete group of collineations. From the main theorem of the author in a preceding paper [same Doklady

(N.S.) **57**, 867–870 (1947); these Rev. **9**, 468] it follows that the order of the complete groups of collineations (in a space X_n with the Weyl tensor $W^{\alpha}_{\beta\gamma\delta}\neq 0$) is not greater than n^2+n. The author gives the proof of the following theorem. The order of the complete group of collineations in the n-dimensional spaces X_n ($n\geqq 3$) with projective connection with the Weyl tensor $W^{\alpha}_{\beta\gamma\delta}\neq 0$ is less than or equal to n^2+n-1. *F. Vyčichlo* (Prague).

Buchdahl, H. A. The Hamiltonian derivatives of a class of fundamental invariants. Quart. J. Math., Oxford Ser. **19**, 150–159 (1948).

The author considers variational (or Hamiltonian) derivatives of invariants which are functions of the metric and curvature tensors of a Riemannian space. An explicit formula is obtained and the well-known divergence relations are verified. *A. Schild* (Toronto, Ont.).

Antosiewicz, H. Über die Anwendungen des Vektorkalküls auf die Geometrie algebraischer Kurven. Monatsh. Math. **52**, 230–247 (1948).

The product of r vectors $v_\alpha{}^i$ ($\alpha=1,\cdots,r$) is called a vector "r-ter Stufe." Its components are defined to be $v_1{}^{(i_1}\cdots v_r{}^{i_r)}$. Let v_α be m vectors. Then the author considers the vector polynomial

$$\prod(v-v_\alpha)\equiv v^m-s_1v^{m-1}+s_2v^{m-2}+\cdots+(-1)^m s_m.$$

The quantity s_r is a vector "r-ter Stufe." It is shown that any symmetrical polynomial $R(v_\alpha{}^i)$ in the components $v_\alpha{}^i$ can be expressed as a polynomial in the components of s_β. Among these components only mn are independent. A set of mn components $s_{\beta i}$ is found such that any rational symmetric function $R(v_\alpha{}^i)$ can be expressed as a rational function of the $s_{\beta i}$. A similar theorem is given for symmetric differentials. An application is found in the theory of algebraic curves. A proof is given of the theorem of Abel. *J. Haantjes* (Leiden).

NUMERICAL AND GRAPHICAL METHODS

Sichel, Herbert S. Fitting growth and frequency curves by the method of frequency moments. J. Roy. Statist. Soc. (N.S.) **110**, 337–347 (1947).

A method of curve fitting based on equating the sum of the nth powers of the observations to an appropriate integral or sum based on the nth power of the ordinate of the fitted curve. The method is applied to growth curves (exponential, Gompertz and logistic) and to frequency curves. *C. P. Winsor* (Baltimore, Md.).

Salzer, Herbert E. An alternative definition of reciprocal differences. Philos. Mag. (7) **39**, 649–656 (1948).

Let $\rho_n(x_1,\cdots,x_n,x_{n+1})$ be Thiele's reciprocal difference for $n+1$ arguments. The author proposes to define a new reciprocal difference by

$$P_n(x_{n+1},x_n,\cdots,x_1)=\rho_n(x_1,\cdots,x_n,x_{n+1})-\rho_{n-2}(x_1,\cdots,x_{n-1}).$$

The new difference is not a symmetric function of the arguments, and the resulting interpolation scheme can be used only for arguments in arithmetical progression. *L. M. Milne-Thomson* (Greenwich).

Woodward, P. M. Tables of interpolation coefficients for use in the complex plane. Philos. Mag. (7) **39**, 594–604 (1948).

If the function $u(x,y)$ satisfies Laplace's equation, it is sufficient for interpolation to give the (2d and 4th) "coupled differences" $\frac{1}{2}(\Delta_x{}^2-\Delta_y{}^2)u$ and $\Delta_x{}^2\Delta_y{}^2u$ in the tables of u. The present table gives the coefficients C_2 and C_4 of the 2d and 4th differences to 8 or 7 decimals. *E. Bodewig.*

Martinotti, Pietro. Interpolazione e medie. Pont. Acad. Sci. Acta **6**, 323–332 (1942).

Étant donné un ensemble de valeurs observées, parmi les méthodes d'interpolation qui se proposent d'en donner une représentation au moyen d'une expression analytique dépendant de paramètres, l'auteur distingue les méthodes directs (exemple: méthode des moments) et indirectes (exemple: méthode des moindres carrés). Pour toutes ces méthodes il transforme les équations en égalités entre moyennes et étand ces égalités à d'autres moyennes que celles qui se présentent immédiatement dans le procédé employé. *J. Favard* (Paris).

Couffignal, Louis. Sur la précision des solutions approchées d'un système d'équations linéaires. C. R. Acad. Sci. Paris **227**, 30–32 (1948).

A simple result concerning the roots of two neighboring systems of equations: $Ax=b$ and $(A+\Delta A)x=b+\Delta b$. *E. Bodewig* (The Hague).

Zurmühl, Rudolf. Runge-Kutta-Verfahren zur numerischen Integration von Differentialgleichungen n-ter Ordnung. Z. Angew. Math. Mech. **28**, 173–182 (1948).

The author considers the general nth order differential equation

$$(1)\qquad y^{(n)}=f(x,y,y',\cdots,y^{(n-1)})$$

and describes a modification of the Runge-Kutta process designed to obtain a higher order of accuracy for the same number of computational steps. As with the Runge-Kutta process, from a given set of derivatives $y^{(0)},y',\cdots,y^{(n-1)}$ at $x=x_0$ those at $x=x_0+\frac{1}{2}h$ and $x=x_0+h$ are first computed from Taylor expansions involving all derivatives up to $y^{(n)}$, with $y^{(n)}$ itself being computed from (1). Substitution of the Taylor expansions in (1) provides approximate values of $y^{(n)}$ at $x_0+\frac{1}{2}h$ and x_0+h. These can then in turn be used to improve the original Taylor expansion of $y,y',\cdots,y^{(n-1)}$ at x_0+h by replacing the last term in these expansions $(y^{(n)}(x_0))$ by a suitably chosen average between $y^{(n)}(x_0)$, $y^{(n)}(x_0+\frac{1}{2}h)$ and $y^{(n)}(x_0+h)$. The author's particular choice of these averages results in an improved accuracy and his method is particularly convenient for $n=4$. Tables facilitating the formation of these averages are provided. The computational lay-out makes use of reduced derivatives and is illustrated with an example ($y''''=y$). The method is laborious, however, if high accuracy is not required. *H. O. Hartley* (London).

Strubecker, Karl. Zur graphischen Integration der linearen Differentialgleichung n. Ordnung $(a_nD+1)(a_{n-1}D+1)\cdots(a_1D+1)y=s(x)$. Arch. Math. **1**, 65–72 (1948).

The author considers the inhomogeneous linear differential equation of the nth order with constant coefficients in the form given in the title, where the operator $D=d/dx$. As a first step a graphical method of solving the first order equation $(aD+1)y=s(x)$ is described: from a given graph

of the input function $s(x)$ a graph of the solution $y(x)$ is constructed by an iterative chord-tangent process which, for the examples considered, appears to be of satisfactory accuracy. A graph of the nth order equation can then be drawn by repeating the above construction n times, the solution of each construction providing the input graph for the next construction. The accumulation of graphical error in this process is not discussed. The method is attractive for the graphical construction of solutions of an aperiodic type but appears to be unsuitable for highly oscillatory problems. A generalisation to cover linear differential equations with variable coefficients is contemplated.

H. O. Hartley (London).

Pflanz, Erwin. Bemerkungen über die Methode von G. Duffing zur Integration von Differentialgleichungen. Z. Angew. Math. Mech. **28**, 167–172 (1948).

The author describes a method of approximate step by step integration of the first order differential equation $y'=f(x,y)$, previously derived by Duffing. The author's derivation of this method makes use only of elementary Taylor expansions and provides gauges of accuracy under the assumption that certain derivatives of f are monotonic. A generalisation to partial differential equations of the first order is given and illustrated with the equation $x^2yz_x+z_y=y^2z$. Numerically the method appears to be more laborious than those directly based on finite difference formulae.

H. O. Hartley (London).

Diaz, J. B., and Greenberg, H. J. Upper and lower bounds for the solution of the first biharmonic boundary value problem. J. Math. Physics **27**, 193–201 (1948).

Let $w(x,y)$ be a solution of the boundary value problem $\Delta\Delta w=p$ in R; $w=f$, $\partial w/\partial n=g$ on C, where R is a plane domain with the boundary C. The authors obtain upper and lower bounds for $w(x_0,y_0)$, the value of w at a point in R, by a method which is applicable to many other problems.

If u is a function satisfying the boundary conditions and v is a function satisfying the partial differential equation, then the authors obtain by applying Green's classical identity and Schwarz's inequality a pair of inequalities of the form $H(u-w)\leqq H(u-v)$, $H(v-w)\leqq H(u-v)$, where $H(\varphi)=\int_R(\Delta\varphi)^2dxdy$.

Together with the function w the authors consider a function $\bar{w}$, the solution of the boundary value problem $\Delta\Delta\bar{w}=0$ in R, $\bar{w}=-r^2l(r)$, $\delta\bar{w}/\delta n=-\partial(r^2lr)/\partial n$ on C, and in analogy with the functions u and v associated with the function w a pair of functions $\bar{u}$ and $\bar{v}$ associated with the function $\bar{w}$. In the expression for $w(x_0,y_0)$ derived from Green's classical identity appears an unknown line integral containing the values of w and $\delta w/\delta n$ on C. But the same line integral appears also in the expressions for $H(u-w,\bar{u}-\bar{w})$, $H(\varphi,\psi)=\int_R\Delta\varphi\Delta\psi dxdy$, to which the above inequalities are applicable.

In this way the authors obtain two inequalities of the form $(8\pi w(x_0,y_0)-b)^2\leqq ac$, $(8\pi w(x_0,y_0)-b')^2\leqq ac$; $a=H(u-v)$, $c=H(\bar{u}-\bar{v})$, where b and b', respectively, are approximate values of $8\pi w(x_0,y_0)$. In order to improve these bounds one may add to u a linear set of functions u_i and to v a linear set of functions v_i and then minimize $H(u-v)$ in order to determine the coefficients of the best linear combinations. If the sequences u_i and v_i are complete in a certain sense defined by the authors the approximations will converge to the value $8\pi w(x_0,y_0)$.

P. Funk (Vienna).

Gilles, D. C. The use of interlacing nets for the application of relaxation methods to problems involving two dependent variables. Proc. Roy. Soc. London. Ser. A. **193**, 407–433 (1948).

The author describes a modification of the relaxation method for the numerical solution of a system of two partial differential equations of the second order for two functions $u(x,y)$ and $v(x,y)$ such as, for instance, the equations of plane strain:

$$2(1-\sigma)u_{xx}+(1-2\sigma)u_{yy}+v_{xy}=0,$$
$$(1-2\sigma)v_{xx}+2(1-\sigma)v_{yy}+u_{xy}=0.$$

The essence of the modification is the use of two interlacing grids, one, $(x,y)=(hi,hj)$, for u and a second, $(x,y)=((i+\frac{1}{2})h,(j+\frac{1}{2})h)$, for v. In most of these problems the equations link even order differentials of u with regard to x (or y) and odd order differentials of v with regard to x (or y) and vice versa. It is therefore clear that if u and v are tabulated at interlacing grids the more accurate formulae of the "midpoint" type can be used, thus enabling the computer to work at a wider grid to the same accuracy. Four examples are treated: Plane strain with boundary displacements given, stresses in a solid of revolution, the conjugate plane harmonic functions, the quasi-plane harmonic equation. Difficulties arise when u and v are prescribed on the same linear boundary as only one of the grids will coincide with this. Thus for one of the functions (say v) the boundary condition must be replaced by a Lagrangean formula to be satisfied by v at certain internal grid points and certain added, "fictitious," points, and this may result in considerable loss of accuracy. The treatment of curved boundaries by the same method is not necessarily less accurate than with a single grid. One misses a discussion of a final check and/or improvement of accuracy by a higher order difference formula.

H. O. Hartley (London).

Tranter, C. J. The combined use of relaxation methods and Fourier transforms in the solution of some three-dimensional boundary value problems. Quart. J. Mech. Appl. Math. **1**, 281–286 (1948).

For those partial differential equations which one wishes to solve over regions for which analytical solutions are not available, the author proposes to take out any known portion of the solution and solve the reduced equation by relaxation methods. As an example the following problem is studied: $\nabla^2V+f(x,y,z)=0$ is to be solved subject to the boundary conditions (a) $V=h_1(x,y)$, $z=0$; (b) $V=h_2(x,y)$, $z=\pi$; (c) $V=g(x,y,z)$ on the surface of a cylinder whose generators are parallel to the z-axis. Here the z dependence is removed and the relaxation method takes over.

A. Heins (Pittsburgh, Pa.).

Korsakov, O. N. Displacement errors of mechanisms with cylindrical gear wheels. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1297–1312 (1948). (Russian)

This paper came out of the error analysis school of Bruevič and harks back to the latter's book "The Precision of Mechanisms" [Tehteoretizdat, 1946]. The procedure of Byhovskiĭ's paper [Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 1455–1512; these Rev. **9**, 536] is applied to the title problem. The variation of the displacement is expressed in terms of seventeen primary variations ("errors") regarded as random quantities with Gaussian or normal

distribution. The expected value and the standard deviation are determined neglecting "second order" quantities. The expected value of the error is zero for a population of mechanisms. The maximum displacement error is also expressed (to the first order), discussed and represented by several graphs. *A. W. Wundheiler* (Chicago, Ill.).

RELATIVITY

Bondi, H. Spherically symmetrical models in general relativity. Monthly Not. Roy. Astr. Soc. **107**, 410–425 (1947).

The author considers the most general space-time which satisfies certain conditions, chief of which are that there is spherical symmetry, that the fundamental world-lines are nonintersecting geodesics, and that pressure is everywhere zero. The corresponding metric is

$$ds^2=dt^2-X^2(r,t)dr^2-Y^2(r,t)(d\theta^2+\sin^2\theta d\phi^2),$$

where $X(r,t)$, $Y(r,t)$ satisfy certain equations resulting from the vanishing of every component of the energy tensor $T^{\mu\nu}$ other than T^{00}. The cosmical constant in the field equations is at first neglected. The world-lines of the particles of the system are the geodesics along which r, θ and ϕ are constant, and it is shown that the function $Y(r,t)$ is closely connected with the astronomical distance (from brightness) of the particle (r,θ,ϕ). The relation between $\partial Y/\partial t$, Y and r derived from the field equations is the equation of radial motion of the particle. This gives the effective gravitating mass, which is found to depend upon the total energy within the spherical shell containing the particle. The equations of motion and the Doppler shift are considered in more detail, and the results are applied to the standard cosmological models. *A. G. Walker* (Sheffield).

Ghosh, J. A type of solutions of Einstein's gravitational equations. Bull. Calcutta Math. Soc. **40**, 45–47 (1948).

Solutions of the gravitational field equations in which the line element has the form $ds^2=e^\nu dt^2-(e^\lambda dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)$ are obtained by making the auxiliary assumption that the stress components are linearly dependent. The solutions are left in the form of infinite series but could have been expressed in terms of the hypergeometric function. There are numerous misprints and errors through the paper. The solution given by equation (24) is incorrect.
M. Wyman (Edmonton, Alta.).

Karmarkar, K. R. On stationary line-elements. Proc. Nat. Inst. Sci. India **13**, 151–155 (1947).

The author obtains necessary and sufficient conditions for a spherically symmetric line element to be stationary.
M. Wyman (Edmonton, Alta.).

Narlikar, V. V., and Vaidya, P. C. Non-static electromagnetic fields with spherical symmetry. Proc. Nat. Inst. Sci. India **14**, 53–54 (1948).

Up to the present time only one solution, with spherical symmetry, of the electromagnetic field equation was known. The authors proceed to find another which is nonstatic in character. *M. Wyman* (Edmonton, Alta.).

Narlikar, V. V., and Singh, K. P. On a gravitational invariant. Proc. Nat. Inst. Sci. India **14**, 121–123 (1948).

The gravitational invariant $B_{ijkl}B^{ijkl}$ is discussed and evaluated for several different line elements.
M. Wyman (Edmonton, Alta.).

Mosharrafa, A. M. The metric of space and mass deficiency. Philos. Mag. (7) **39**, 728–738 (1948).

The first half is a repetition of an earlier paper of the author's [Proc. Math. Phys. Soc. Egypt **3** (1945), 19–24 (1946); these Rev. **8**, 412]. The author's theory is applied to the problem of nuclear mass defects. *A. Schild.*

MECHANICS

Hydrodynamics, Aerodynamics

Caldonazzo, Bruto. Sui moti liberi di un mezzo continuo. Ann. Mat. Pura Appl. (4) **26**, 43–55 (1947).

A study of those motions of a fluid in which each particle moves in a straight line with constant velocity, in particular, homographic motions, in which the initial velocity is a linear function of position, and stationary motions, in which the trajectories coincide with the lines of flow.
J. L. Synge (Dublin).

García, Godofredo. Sur une formule exacte, cardinale et canonique des tensions internes et sur l'équation cardinale, canonique du mouvement des fluides visqueux. Ann. Soc. Polon. Math. **21**, 107–113 (1948).

The two-dimensional formulation of equations given by the author [Actas Acad. Ci. Lima **10**, 117–170 (1947); these Rev. **9**, 475]. The notations are not clear.
L. M. Milne-Thomson (Greenwich).

Truesdell, C. On the differential equations of slip flow. Proc. Nat. Acad. Sci. U. S. A. **34**, 342–347 (1948).

What the author seems to aim at in this paper is a very comprehensive theory of the motion of a viscous compressible fluid, which may be a mixture of several substances and anisotropic. The argument is condensed and on that account difficult to follow. Contrary to the implication of the title, differential equations of motion are not written down, and the discussion centers on a formula expressing stress as a power series in the viscosity μ, the coefficients up to that of μ^2 being given explicitly for an isotropic fluid. These coefficients are functions of a considerable number of variables which include temperature, mean pressure p_m, gradient of pressure p (p is defined thermodynamically), gradient of body force, the rate of deformation tensor, the vorticity tensor, and higher derivatives of these quantities. The author draws attention to one rather startling and physically improbable consequence of ~~his theory,~~ namely that a mass of fluid rotating as a rigid body experiences a stress which depends on the viscosity. ~~A fuller account of the theory is to appear elsewhere.~~ *J. L. Synge.*

Rivlin, R. S. The hydrodynamics of non-Newtonian fluids. II. Proc. Cambridge Philos. Soc. **45**, 88–91 (1949).

In part I [Proc. Roy. Soc. London. Ser. A. **193**, 260–281 (1948); these Rev. **10**, 73] the author established general equations of motion for a non-Newtonian fluid. In this paper these equations are applied to the flow of such a fluid

... of the dependence of stress on vorticity ....

He therefore suggests that vorticity should not be included in the expressions for stress.

Errata, p. 356

through a cylindrical tube of circular cross section. It is found that the axial normal stress σ_{zz} is no longer uniformly distributed over the cross section of the tube, as in the case of a Newtonian fluid. However, the derivative $\partial\sigma_{zz}/\partial z$ is constant throughout the tube. *W. Prager.*

Holt, M. The behaviour of the velocity along a straight characteristic in steady irrotational isentropic flow with axial symmetry. Quart. J. Mech. Appl. Math. 1, 358–364 (1948).

The author investigates the behavior of the velocity along a straight characteristic in steady irrotational isentropic flow with axial symmetry. If ϵ is the angle between the straight characteristic and the axis, μ the Mach angle, or the angle between the velocity vector and the characteristic, the different flow patterns are classified according to whether ϵ, μ are positive or negative and whether the characteristic reaches the axis. The author obtains the following result. If (1) $\epsilon>0$, $\mu>0$; (2) $\epsilon<0$, $\mu>0$ and axis included; (3) $\epsilon>0$, $\mu<0$ and axis excluded, the solutions have limiting lines where the acceleration is infinite, and beyond which the potential flow cannot be continued. *H. S. Tsien.*

Sneddon, I. N., and Fulton, J. The irrotational flow of a perfect fluid past two spheres. Proc. Cambridge Philos. Soc. 45, 81–87 (1949).

The authors treat the boundary value problem of an irrotational flow past two spheres of an incompressible non-viscous fluid. This problem had been considered previously in both electrostatics and hydrodynamics. Solutions had been obtained but all in cumbersome form. The object of this paper is to show that by the use of a general formula for the velocity potential due to Weiss [same Proc. 40, 259–261 (1944); these Rev. 6, 191] a simple closed expression may be obtained for the solution of the simplest boundary value problem involving two spheres. It is assumed that the line of centers of the spheres is parallel to the undisturbed uniform flow. *A. Gelbart* (Syracuse, N. Y.).

Shen, Yuan. The flow of a compressible fluid past quasi-elliptic cylinders at high subsonic speeds. Sci. Rep. Nat. Tsing Hua Univ. 5, 29–51 (1948).

The paper deals with the problem of constructing a solution for the flow of a compressible fluid past quasi-elliptic cylinders by the hodograph method. By starting with two power series representing the stream function of the incompressible flow past an elliptic cylinder, in the respective domains of convergence in the hodograph plane, the author constructs a corresponding solution for the compressible fluid by simply replacing the proper elementary solutions. The solutions thus obtained do not join to each other smoothly on the circle of convergence and consequently fail to represent the same flow. As an attempt to simplify the numerical calculation, the hypergeometric functions are approximated by $r(\tau)\{S(\tau)\}^n$ for the subsonic range, where τ is proportional to the square of the local flow speed. The forms of the functions $r(\tau)$ and $S(\tau)$ are not given explicitly, but it is indicated that they are determined numerically. Several numerical examples have been treated by this method, and the results are presented in tabular and graphical form.

In an appendix, the author makes brief reference to the work of Tsien and Kuo [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 995 (1946); these Rev. 8, 237]. He points out that their method takes account of the continuity conditions at the circle of convergence, while his does not, and that there are other differences. Nevertheless, he believes that his numerical results are near to theirs. *W. R. Sears* (Ithaca, N. Y.).

Frankl', F. I. Asymptotic expansion of Čaplygin's functions. Doklady Akad. Nauk SSSR (N.S.) 58, 757–760 (1947). (Russian)

S. A. Chaplygin [Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz.-Mat. 21, 1–121 (1904); English translation in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1063 (1944); these Rev. 7, 495] introduced the solutions $\psi_n=z_n(\tau)\sin 2n\theta$ of the gas-dynamical equation

$$\frac{\partial}{\partial\tau}\left[\frac{2\tau}{(1-\tau)^\beta}\frac{\partial\psi}{\partial\tau}\right]+\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial^2\psi}{\partial\theta^2}=0,$$

where $\tau=w^2/w_{\max}^2$, w is the speed, $w_{\max}$ the maximum speed, $\beta=1/(\gamma-1)$, and $\gamma=c_p/c_v$. In the discussion of flows which are both subsonic and supersonic, the ratio $z_n(\tau)/z_n(\tau^*)$ is of interest (here $\tau^*=(\gamma-1)/(\gamma+1)$). In a previous paper [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 387–422 (1945); these Rev. 8, 416] the author has given the following asymptotic expansion for $z_n'(\tau^*)/z_n(\tau^*)$:

$$z_n'(\tau^*)/z_n(\tau^*)=n^{\frac{2}{3}}(A_0+A_1n^{-\frac{2}{3}}+\cdots+A_kn^{-2k/3})+O(1)n^{-2k/3}.$$

In the present paper he strengthens his previous result, and shows that

$$z_{n/2}(\tau)/z_{n/2}(\tau^*)=\zeta_n(\eta)=\lambda(n^{\frac{2}{3}}\eta)+\lambda^{(1)}(n^{\frac{2}{3}}\eta)n^{-\frac{2}{3}}+\cdots$$
$$+\lambda^{(k)}(n^{\frac{2}{3}}\eta)n^{-2k/3}+\Delta_n^{(k+1)}(n^{\frac{2}{3}}\eta)n^{-2(k+1)/3}$$

with the following bounds for the remainder term and its derivative:

$$|\Delta_n^{(k+1)}(\xi)|<A^{(k+1)},\quad |\Delta_n^{(k+1)\prime}(\xi)|<A^{(k+1)}.$$

Here

$$\eta=\left[\frac{3}{4}\int_\tau^{\tau^*}(1-\tau/\tau^*)^{\frac{1}{2}}(1-\tau)^{-\frac{1}{2}}\tau^{-1}d\tau\right]^{\frac{2}{3}}$$

and the expansion is valid for $\eta\geqq 0$ $(0\leqq\tau\leqq\tau^*)$. The functions $\zeta_n(\eta)$ are defined by the conditions

$$\zeta_n''(\eta)+b(\eta)\zeta_n'(\eta)-n^2\eta\zeta_n(\eta)=0,\quad \zeta_n(0)=1,\quad \zeta_n(+\infty)=0,$$

where

$$b(\eta)=\tfrac{1}{2}(d/d\eta)\log(K/\eta)=b_0+b_1\eta+b_2\eta^2+\cdots,$$

$K=(1-\tau/\tau^*)/(1-\tau)^{1/\tau^*}$. The function $\lambda(\xi)$ is Airy's function, defined by the conditions $\lambda''(\xi)-\xi\lambda(\xi)=0$, $\lambda(0)=1$, $\lambda(+\infty)=0$, and the functions $\lambda^{(i)}(\xi)$ are defined by $\lambda^{(i)\prime\prime}(\xi)-\xi\lambda^{(i)}(\xi)=q_i(\xi)$, $\lambda^{(i)}(0)=\lambda^{(i)}(+\infty)=0$, where

$$q_i(\xi)=-\{b_{i-1}\xi^{i-1}\lambda'(\xi)+b_{i-2}\xi^{i-2}\lambda^{(1)\prime}(\xi)+\cdots+b_0\lambda^{(i-1)\prime}(\xi)\}.$$

J. B. Diaz (Providence, R. I.).

Munk, Max M., and Prim, Robert C. On the canonical form of the equations of steady motion of a perfect gas. J. Appl. Phys. 19, 957–958 (1948).

By using the reduced velocity of the fluid, which is the ratio of the fluid velocity to the maximum velocity along any stream line, the authors decrease the number of basic equations for the steady motion of a perfect gas from five to four. These the authors call the canonical form. This result was previously obtained by B. Hicks, P. Guenther and R. Wassermann [Quart. Appl. Math. 5, 357–361 (1947); these Rev. 9, 112]. *H. S. Tsien* (Cambridge, Mass.).

Manwell, A. R. The analysis of subsonic flow and constant velocity aerofoils. Philos. Mag. (7) **39**, 712–722 (1948).

The author points out that by a change of scale the differential equations for compressible flow in the hodograph plane can be considered as those of incompressible flow in a shallow dish with the depth of fluid specified as a function of the distance from the center of the dish. In the case of the Kármán-Tsien approximation, this depth of fluid is a constant and thus the method of complex variables can be applied. The author further suggests that the boundary of the profile be specified as a line in the hodograph plane so as to simplify the mathematical problem. He then uses this approach to obtain the solution in closed form for a family of symmetrical profiles at zero angle of attack with almost constant velocity over the surface. For small thickness ratios, his profile is very similar to an ellipse of the same thickness ratio. *H. S. Tsien* (Cambridge, Mass.).

Ferrari, Carlo. Sulla determinazione del flusso attraverso ad una schiera di profili alari con forte curvatura. Aerotecnica **28**, 119–135 (1948).

Methods are given for determining the flow of an inviscid fluid, at low or at most subsonic speeds, past a cascade of highly cambered aerofoils. At low speeds the region of flow is conformally transformed, by prescribed elementary functions, into the region outside an approximate circle, which is then transformed into that outside a more exact circle by a Fourier series method. The flow is thus obtained for a number of interesting examples. The work is extended to subsonic high speed flow by use of an elaboration of the Kármán-Tsien method [von Kármán, J. Aeronaut. Sci. **8**, 337–356 (1941); these Rev. **3**, 220] based on work by Germain [C. R. Acad. Sci. Paris **223**, 532–534 (1946); these Rev. **8**, 237]. *M. J. Lighthill* (Manchester).

Êfros, D. A. The calculation of the hydrodynamical forces acting on a cavitational contour in a plane-parallel flow. Doklady Akad. Nauk SSSR (N.S.) **60**, 29–31 (1948). (Russian)

In an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 267–270 (1946); these Rev. **8**, 105] the author has introduced a particular mathematical model, sometimes known as the re-entrant jet model, for an attached cavitation bubble behind an obstruction in the form of an arc in a two-dimensional flow. This model was apparently first introduced by H. Wagner and has been studied by G. Kreisel, D. Gilbarg and D. H. Rock, M. I. Gurevich and others. The re-entrant jet, which has been observed in experiment and which, if it does not first degenerate in some manner, will strike the back of the obstruction, is in the mathematical model conveniently removed onto a second sheet of a Riemann surface. In the author's earlier paper he studied the conformal mapping problem associated with this model and, in order to define the solution uniquely, assumed that there was no circulation along a contour surrounding obstruction and bubble. In the present paper he allows circulation, assumes the lift is given entirely by the circulation according to the Kutta-Joukowsky formula, and is again able to uniquely define the solution, so that both lift and drag may be determined. *J. V. Wehausen.*

de Kármán, Théodore. Sur la théorie statistique de la turbulence. C. R. Acad. Sci. Paris **226**, 2108–2111 (1948).

The author considers the three components u_i of the turbulent velocity fluctuations of an incompressible viscous fluid (homogeneous and isotropic turbulence); using three-dimensional harmonic analysis it is assumed that for each component:

$$u_i=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(t, k_1, k_2, k_3)e^{-i(k_1x_1+k_2x_2+k_3x_3)}dk_1dk_2dk_3.$$

According to this assumption, $F(k)dk$ is the part of the mean kinetic energy $\frac{1}{2}\sum\overline{u_i^2}$ resulting from the harmonic components of the u_i for which $k<\sum\sqrt{k_i^2}<k+dk$. Thus, from the energy equation for a viscous fluid one gets the fundamental equation

$$\text{(I)}\quad \partial F/\partial t+\int_0^{\infty}\Theta[F(k), F(k'), k, k']dk'=-2\nu k^2F(k),$$

where Θ is the energy transferred in the fluid from harmonic components for which $\sqrt{k_i^2}$ is between k' and $k'+dk'$ to harmonic components for which $\sqrt{k_i^2}$ is between k and $k+dk$. Very little is known about this function Θ; the result has been given by Kolmogoroff, Obhukoff, Weiszäcker, Heisenberg and Onsager that $F(k)=Ak^{-5/3}$ is a particular solution of equation (I), if one assumes that Θ is a monomial function $\Theta=-CF(k)^{\alpha}F(k')^{\alpha'}k^{\beta}k'^{\beta'}$. In the case of large Reynolds numbers the author gives explicit formulas for the correlation functions f and g, which are said to be in good agreement with measurements made by J. Lauffer at Pasadena. *J. Kampé de Fériet* (Lille).

Bondi, H. The growth of meteorological disturbances. Proc. Cambridge Philos. Soc. **45**, 92–98 (1949).

The motions of a gas slightly disturbed from equilibrium under the influence of gravity are considered. Heat conduction and viscosity are at first neglected. The well-known sharp distinction between slow large-scale (meteorological) and fast small-scale (acoustical) phenomena is confirmed by the mathematical analysis. Only the former motions are considered here, and the author confines himself to those disturbances which involve regions in which the percentual change of the temperature is small. For the stable case the result of the disturbance will be a slow oscillation while in the unstable case if the lapse rate exceeds the adiabatic by 1% the strength of the disturbance will be multiplied by 10,000 within $1\frac{1}{3}$ hours. If the effects of viscosity and heat conduction are considered in a semi-empirical fashion it can be concluded that because of the increase of the dissipating effects with decreasing magnitude of the disturbance a limiting size must exist below which the disturbance cannot grow, even though the stratification is unstable. If the lapse rate exceeds the adiabatic by 1% this limit would be represented by the diameter 230 m. The effect of the earth's rotation is negligible. *B. Haurwitz* (New York, N. Y.).

Elasticity, Plasticity

Rivlin, R. S. A uniqueness theorem in the theory of highly-elastic materials. Proc. Cambridge Philos. Soc. **44**, 595–597 (1948).

The stored-energy function of an incompressible neo-Hookean material [cf. Philos. Trans. Roy. Soc. London. Ser. A. **240**, 459–490 (1948); these Rev. **10**, 168] is here assumed to be of the form $W=C_1(I_1-3)+C_2(I_2-3)$, where C_1 and C_2 are positive constants characterizing the material and $I_1=\lambda_1^2+\lambda_2^2+\lambda_3^2$, $I_2=\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2+\lambda_1^2\lambda_2^2$, the λ's being

the principal extensions at the point considered. It is proved that, if a unit cube of the material is subjected to the action of equal and oppositely directed forces acting normally on its faces, the finite deformation produced is uniquely determined, provided that the forces per unit area, measured in the deformed state, are specified. It is further proved that the deformed state is stable. *J. L. Synge* (Dublin).

Gol'denblat, I. I. On a method in the theory of elastic and plastic deformations. Doklady Akad. Nauk SSSR (N.S.) 61, 1001–1004 (1948). (Russian)

The author claims that the basic thermodynamical relations for reversible isothermal deformations of a solid may be applied to (irreversible) plastic deformations too, as long as unloading is avoided. Assuming the free energy to be a function of the linear and quadratic invariants of the tensors of stress and strain, he obtains stress-strain relations for plastic materials with or without strain-hardening.

W. Prager (Providence, R. I.).

Colonnetti, G. Saggio di impostazione generale del problema delle deformazioni viscose. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 515–519 (1948).

The paper is concerned with creep in an elastic-plastic solid. The initial stresses and permanent strains which represent the instantaneous effect of a given loading schedule are supposed to be known, and it is assumed that slow inelastic deformations will then occur under constant surface tractions, the inelastic velocity strain being proportional to the stress. It is shown that this inelastic deformation may be considered as the result of the superposition of two special inelastic deformations, the first of which leaves the stresses and hence the elastic deformation unchanged while the second is accompanied by an elastic deformation of equal magnitude and opposite sign. The author then discusses the circumstances under which only one of these special inelastic deformations occur: the inelastic deformation will be of the first kind if the permanent strains vanish (creep in an elastic solid) and of the second kind if the surface tractions vanish (creep in an elastic-plastic body with residual stresses).

W. Prager (Providence, R. I.).

Gross, B. On creep and relaxation. II. J. Appl. Phys. 19, 257–264 (1948).

Paper I [same J. 18, 212–221 (1947); these Rev. 9, 546] dealt with the transient response of a linear viscoelastic medium, resulting from the sudden application of a constant load or a constant deformation. The present paper develops the theory of the steady-state behavior under alternating load and deformation. Relations are established between the loss factor, the storage factor, the distribution functions (of elastic relaxation times or elastic orientation times), and the Laplace transforms of the creep function and the relaxation function. *T. Alfrey, Jr.* (Brooklyn, N. Y.).

Burgers, J. M. Non-linear relations between viscous stresses and instantaneous rate of deformation as a consequence of slow relaxation. Nederl. Akad. Wetensch., Proc. 51, 787–792 (1948). (English. Esperanto summary)

The physical concepts involved in the deformation of visco-elastic materials are considered, providing an explanation of the coexistence of a constant elastic deformation and a continuously progressing deformation as found in laminar flow. The viscous stresses are considered to be due to deviation of the equilibrium pattern of the molecular structure caused by the flow. This deviation is being continuously relaxed, and for short relaxation times the viscous stresses are proportional to the rate of strain tensor. For long relaxation times, however, the viscous stresses are influenced by the more remote past, which may cause a deviation in stress direction. A simple example is given.

E. H. Lee (Providence, R. I.).

Arf, Cahit. Sur la détermination des états d'équilibre d'un milieu élastique plan admettant des frontières libres à tensions constantes. Rev. Fac. Sci. Univ. Istanbul (A) 12, 309–344 (1947). (French. Turkish summary)

Consider a plane elastic medium in equilibrium, body forces being absent. If $z=x+iy$, then the components σ_x, σ_y and τ of the stress tensor are known to be given by

$$\begin{aligned}\sigma_x&=\Re[2f(z)+\psi(z)-\bar{z}f'(z)],\\ \sigma_y&=\Re[2f(z)-\psi(z)+\bar{z}f'(z)],\\ \tau&=\Im[-\psi(z)+\bar{z}f'(z)],\end{aligned}$$

where $f(z)$ and $\psi(z)$ are analytic functions. The author proposes to discuss in a series of papers the problem of determining the functions $f(z)$ and $\psi(z)$ defined on plane domains with free boundaries, i.e., such that $4\Re f(z)=\sigma_x+\sigma_y=4\alpha$, where α is a constant. The entire discussion in this first paper is limited to the case when $f(z)$ reduces to a constant α over the whole plane, and not merely on the free boundaries. Under this assumption, the paper contains the solution of the following special problem (A): to determine the free boundaries L of plane simply connected domains C which satisfy the following conditions: (a) C does not contain ∞ and as z tends to ∞ alone a continuous path lying in C, then $\arg z$ tends to one of the n distinct real numbers $\varphi_1, \cdots, \varphi_n$; (b) to each φ_j corresponds a finite set of disjoint intervals $(a_1^{(j)}, b_1^{(j)}), \cdots, (a_{\nu_j}^{(j)}, b_{\nu_j}^{(j)})$, consisting of the limiting values of $\Im[\exp(-i\varphi_j)]$ as z tends to infinity along a continuous path lying in C on which $\arg z\rightarrow\varphi_j$; (c) L possesses at most a finite number of inflection points and cusps; (d) the tensor $(\sigma_x, \sigma_y, \tau)$, i.e., the analytic function $(4\alpha)^{-1}(\sigma_x-\sigma_y-2i\tau)$, has a definite value (finite or infinite) at each point of $C+L-\infty$, and is infinite at at most a finite set of points of C. It is found that a solution of problem (A) can be given in terms of each analytic function $F(T)$ defined in the half plane $\Im T\geqq 0$ whose derivative $F'(T)$ is a rational function satisfying certain conditions (which cannot be enumerated here) and that all solutions of problem (A) fall in this category.

J. B. Diaz (Providence, R. I.).

Sneddon, Ian N. Boussinesq's problem for a rigid cone. Proc. Cambridge Philos. Soc. 44, 492–507 (1948).

The author gives a detailed account of the stress-distribution in a semi-infinite isotropic elastic solid, deformed by a rigid cone. The author's technique is based on a method of integral transforms developed by J. W. Harding and I. N. Sneddon [same Proc. 41, 16–26 (1945); these Rev. 6, 251]. Based on these previous results, formulas, tables, and graphs for the stress-distribution at any interior point of the elastic medium are given in terms of dimensionless parameters. The principal shearing stress is also studied. The work done in pushing a conical punch of given dimensions into the material is also computed. The author points out that this result can be extended to the plastic domain provided that the material is assumed to behave in the manner of a Hencky body. This last restriction should be kept in mind in em-

ploying the author's results for states of stress beyond the elastic limit. *G. H. Handelman* (Pittsburgh, Pa.).

Chien, Wei Zang. Asymptotic behavior of a thin clamped circular plate under uniform normal pressure at very large deflection. Sci. Rep. Nat. Tsing Hua Univ. **5**, 71–94 (1948).

A perturbation method is employed in which the essential quantities are expressed in power series of a small parameter and the resulting sequence of linear differential equations which arise from the von Kármán system are solved. The first problem treated is the circular membrane under uniform normal loading and the author corrects an error in Hencky's original solution. The formulation of the thin circular plate with clamped edges is based upon the membrane solution together with a correction of the edge effect in asymptotic form. The theoretical results are in good agreement with known experimental data. *D. L. Holl.*

Lehnickiĭ, S. G. The bending of a rectangular orthotropic plate resting on parallel rigid ribs. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 339–344 (1948). (Russian)

This note deals with the formulation of the boundary-value problem for small deflections of thin rectangular orthotropic elastic plates reinforced by rigid ribs parallel to one of the sides of the plate. The load is normal to the middle plane, which is regarded as one of the planes of elastic symmetry. The other planes of elastic symmetry are parallel to the sides of the plate. It is shown that the problem is equivalent to an analysis of deflection of a beam resting on an elastic foundation and subjected to tensile stresses along the axis of a beam. The considerations of this paper are based, to a large extent, on the author's book "Anistropic Plates" [Gostehizdat, 1947]. For the solution of an analogous problem for an isotropic plate see A. S. Lokshin [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] (1) **2**, 225–240 (1935)] and A. P. Philippow [ibid. (N.S.) **1**, 187–204 (1937)]. *I. S. Sokolnikoff.*

Green, A. E., and Willmore, T. J. Three-dimensional stress systems in isotropic plates. II. Proc. Roy. Soc. London. Ser. A. **193**, 229–248 (1948).

The problem treated is that of an infinite plate of finite thickness subject to a concentrated force uniformly distributed along a transverse fiber of the plate and acting parallel to the faces of the plate, which are themselves free from stress. The exact solution for the stresses in cylindrical coordinates ρ, θ, ζ is found as a sum of the plane strain solution given by Love, which satisfies all but the conditions of zero normal stress on the faces of the plate, and the solution of the residual problem thereby defined. To obtain the latter solution, the authors make use of a representation given by J. Dougall [Trans. Roy. Soc. Edinburgh **41**, 129–228 (1904)] for displacements in plane plates in terms of two arbitrary harmonic functions. The harmonic functions used here are

$$\omega=\cos\theta\int_0^\infty f(u)J_1(u\rho)\cosh u\zeta du,$$

$$\phi=\cos\theta\int_0^\infty g(u)J_1(u\rho)\cosh u\zeta du,$$

where J_1 is the Bessel function, and f and g are chosen so as to satisfy boundary conditions. Expressions for the average stresses are tabulated and graphs are given for comparison with the results of the approximate theory of generalized plane stress. With certain exceptions the generalized plane stress solution is found to furnish a good approximation to the true stress averages computed. It is noted that the deviations increase with increasing Poisson's ratio. Actual stress values on the faces of the plate and in the middle plane are computed for comparison with the average values. The solution in integral form for the case of a plate of semi-infinite thickness is included.

H. J. Greenberg (Pittsburgh, Pa.).

Panov, D. Yu., and Feodos'ev, V. I. On the equilibrium and instability of sloping shells with large deflections. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 389–406 (1948). (Russian)

The author follows the standard procedure to derive a system of 14 equations in 14 unknowns characterizing the state of equilibrium of thin shallow asymmetric shells under large deflections. Previous formulations of similar problems were concerned with flat plates and with special cases of shells of revolution subjected to loading which preserves spherical symmetry. The resulting equations are too lengthy to be written out in this review. They appear in the paper as 5 equations numbered (2.1), 3 equations (3.3), (3.6), (3.8), and 6 equations (4.3). These equations lead to the equation for the deflection w in the anticipated form

$$-p-D\nabla^4 w+T_x\frac{\partial^2(w_0+w)}{\partial x^2}+2Q\frac{\partial^2(w_0+w)}{\partial y\partial x}+T_y\frac{\partial^2(w_0+w)}{\partial y^2}=0,$$

where $w=w_0(x, y)$ is the equation of the middle surface before deformation, T_x and T_y are the compressive forces, Q is the shearing force, p the external load and D the flexural rigidity.

As an illustration of the solution of this equation the author considers a uniformly loaded circular plate so freely clamped along the contour as to permit free displacements in the radial and tangential directions (i.e., $T=Q=0$ on the contour) and to preclude the angular displacements $\theta=-\partial w/\partial x$ and $\chi=-\partial w/\partial y$. This mode of clamping admits the formation of folds which destroy the circular symmetry. The solution of this problem [pp. 396–406] is carried out by Galerkin's method. As one would expect, the computations leading to the determination of the amplitude of the folds and for the instability criteria are very heavy.

I. S. Sokolnikoff (Los Angeles, Calif.).

Wittrick, W. H. Preliminary analysis of a highly swept cylindrical tube under torsion and bending. Commonwealth of Australia. Council Sci. Ind. Res. Aeronaut. Research Rep. no. **ACA-39**, 23 pp. (1948).

Müller-Magyari, F. Kritische Spannungen dünnwandiger Plattenwerke unter zentrischem Druck. Österreich. Ing.-Arch. **2**, 331–346 (1948).

Consideration is given to local buckling of thin-walled axially loaded columns, like channels and box girders, composed of joined thin rectangular plates each of which is uniformly loaded. The basic solution is for a rectangular plate uniformly loaded along two opposite simply supported edges and elastically constrained along the other two. Results are presented graphically for both negative and positive elastic constraint. Using the curves, an approximate method based on a relaxation of continuity conditions between the plates gives lower bounds for the buckling load. Rectangular and triangular cross-sections are used for illustration. *D. C. Drucker* (Providence, R. I.).

Reissner, Eric. Note on the method of complementary energy. J. Math. Physics 27, 159–160 (1948).

The author considers the forced oscillations of an elastic body (in general anisotropic) under the influence of prescribed surface stresses and body forces, both of these having a simple harmonic time-dependence (time-factor $\cos \omega t$). He considers a class C of states in which the stress and the displacement have this time-factor and are otherwise arbitrary except that (a) the equations of motion are satisfied, and (b) the boundary conditions are satisfied. In general the equations of compatibility are not satisfied, so that stress and displacement are independent. A strain-energy function of position W is defined by the usual quadratic expression in terms of stress, but with the time-factors omitted, and a function of position K is defined by $K=\frac{1}{2}\rho\omega^2(u^2+v^2+w^2)$, where ρ is density and (u, v, w) the displacement with the time-factor omitted. The integral $\int(W-K)dV$, taken throughout the body, is considered for the class of states C, and it is proved that this integral has a stationary value when the equations of compatibility are satisfied by the stress and the displacement is that which corresponds through the expressions for strain and the stress-strain relations, i.e., when the state is the actual state of vibration. *J. L. Synge* (Dublin).

Bahšiyan, F. A. Finite displacements in a hollow sphere subjected to internal pressure. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 137–140 (1948). (Russian)

Formulae for stress and displacement distributions are obtained for a hollow sphere subjected to internal pressure assuming finite strains. A deformation type of stress-strain relationship is used with plastic deformation occurring at a constant value of the second power invariant of the stress deviator (von Mises criterion) and a linear relationship between the stress and strain deviators. However, because of the symmetry of the stress system considered, the reviewer believes that the analysis also applies to a flow type of law. The equilibrium equation and the compressibility relationship are expressed for large radial displacement u; and the Lagrange type of strain components, du/dr and u/r, are used in both the elastic and plastic stress-strain relationships. The solution is given when the sphere is completely elastic, and when it is just completely plastic with the outside layers strained to the yield point strain. For the partly plastic condition incompressibility is assumed in both the elastic and plastic regions. A limit is determined for the internal pressure required to produce flow in a spherical shell. *E. H. Lee* (Providence, R. I.).

Bahšiyan, F. A. An elastic-plastic spherical wave of loading. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 281–286 (1948). (Russian)

The stress waves resulting from a suddenly applied increasing pressure inside a spherical hole in an elastic-plastic material are investigated. The pressure is assumed to be sufficiently large to cause plastic flow immediately, and the problem is considered before other boundaries are reached by the stress waves. A deformation type of stress-strain relationship is adopted, which assumes that the second power invariant of the stress deviator is a function of the second power invariant of the total strain deviator, and that the stress and total strain deviators are linearly related. However, because of the symmetry of the stress system, the reviewer believes that the analysis also applies to a more general flow type of stress-strain relationship.

The equation of motion and the stress-strain relationship determine a quasi-linear hyperbolic differential equation for the displacement, assuming small strains. The solution is evaluated for a material with constant flow stress. A change of dependent variable transforms the equation into the simple wave equation, with the wave velocity in the plastic region less than for the elastic region. The solution is treated separately in the two regions, with continuity conditions imposed at the common boundary. *E. H. Lee.*

Sokolovskiĭ, V. V. The propagation of elastic-viscous-plastic waves in bars. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 261–280 (1948). (Russian)

The theory of the propagation of stress waves in rods is developed for materials with stress-strain relations of the form $Ed\epsilon/dt=d\sigma/dt+\kappa kF(|\sigma|-\sigma_s)$, where F is a monotonically increasing function for positive argument, and zero for negative and zero argument, k is a constant, and $\kappa=\operatorname{sgn}\sigma$. Two uniform stress problems are first considered: a constant strain rate test and the effect of a triangular pulse of stress.

The equations of motion for a rod of varying section are shown to reduce to a pair of linear hyperbolic partial differential equations in the dependent variables σ and v (velocity). The characteristics in the (x, t)-plane are the same as for elastic waves; the differential relations along them are developed. A shock front of stress discontinuity is also shown to be propagated with the elastic wave velocity, and the Hugoniot relations across such a shock are stated. The variation of a stress discontinuity as it propagates is considered, using the Hugoniot relations across the shock and the relationship imposed along the characteristic coincident with the shock. The stress variation is thus determined by an ordinary differential equation, the solution being given for particular cases of cylindrical and conical rods.

Three illustrative problems are given for the special case in which F is a linear function, the stress, velocity, and strain distributions being determined by numerical integration along the characteristics: (1) a pulse of constant stress on the end of a semi-infinite cylindrical rod; (2) a triangular pulse of stress, with an instantaneous rise to the maximum value, acting on the end of a semi-infinite conical rod; (3) a finite cylindrical rod fixed at one end and free at the other, subjected to a suddenly applied constant stress at the free end. *E. H. Lee* (Providence, R. I.).

Sokolovskiĭ, V. V. The propagation of elastic-viscous-plastic waves in bars. Doklady Akad. Nauk SSSR (N.S.) 60, 775–778 (1948). (Russian)

This paper is an abridged version of that reviewed above. *E. H. Lee* (Providence, R. I.).

Rahmatulin, H. A., and Šapiro, G. S. On the propagation of plane elastic-plastic waves. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 369–374 (1948). (Russian)

This paper considers the longitudinal wave propagation along a semi-infinite cylindrical rod subjected to a pulse of pressure on the end. The pressure is considered to be of sufficient magnitude to cause plastic flow, and a linear work-hardening stress-strain relationship is assumed. Two particular problems are considered: gradual increase and decrease of pressure, with the maximum value held for a finite time; discontinuous rise and fall of pressure. The first problem is treated by the usual method of characteristics

for the resulting wave equation. The second involves only waves of stress discontinuity, and the conditions across them are obtained from momentum change considerations, which determine the development of the wave system.

E. H. Lee (Providence, R. I.).

Föppl, O. Biegeschwingungen einer Welle, die masselose Trägheitsmomente trägt. Z. Angew. Math. Mech. **24**, 204–209 (1944).

The author discusses the vibrations of a shaft which carries at one point an object whose mass is neglected but whose moment of inertia is taken into consideration.

G. F. Carrier (Providence, R. I.).

Mudrak, Walter. Bestimmung der Eigenschwingungszahlen von durchlaufenden Trägern und Rahmen. Z. Angew. Math. Mech. **28**, 258–263 (1948). (German. Russian summary)

Für die Berechnung der Eigenschwingungszahlen eines an beiden Enden unverschieblich gelagerten, aber elastisch eingespannten Stabes von konstantem Querschnitt und gleichförmiger Massebelegung wird ein Diagramm gegeben. Seine Anwendung zur Berechnung der Eigenschwingungszahlen von durchlaufenden Trägern und von Rahmen, deren Felder die obigen Bedingungen erfüllen, wird gezeigt und an Beispielen erläutert. *Author's summary.*

Sommerfeld, A. Berichtigungen und Ergänzungen zu der Arbeit: Die frei schwingende Kolbenmembran. Ann. Physik (6) **2**, 85–86 (1948).

A numerical slip in the value of a coefficient is corrected [same Ann. (5) **42**, 389–420 (1943); these Rev. **5**, 121]. The series expression for d_n is summed. *D. G. Bourgin.*

Federhofer, K. Berechnung der Grundschwingzahl der gleichmässig belasteten dünnen Kreisplatte mit grosser Ausbiegung. Österreich. Ing.-Arch. **2**, 325–331 (1948).

Consider a circular plate with built-in edge under uniform load of appreciable magnitude, so that the plate deflections exceed the allowable limits of the Kirchoff small-bending theory. The present paper studies the fundamental natural frequency of such a plate executing vibrations of small amplitude about its equilibrium (static) configuration. In essence, the effective plate stiffness increases with load, due mainly to the appearance of significant membrane stresses, so that the natural frequency of plate vibration also increases. The author derives an approximate expression for the static plate deflection under load, based on previous work of the author [Luftfahrtforschung **21**, 1–10 (1944); these Rev. **6**, 28] and A. Nadai [Die elastischen Platten, Springer, Berlin, 1925, p. 288]. For the vibration problem a small departure from the equilibrium position is considered, the mode shape being taken the same as that of the static deflection. A straightforward energy calculation is then employed to derive the fundamental natural vibrational frequency.

Typical of the results is the case where the static deflection at the plate center is 10.9 times as large as the plate thickness; compared with the results from the small-bending theory, the fundamental plate frequency is increased by about 13 times. The results of the present analysis are compared with those obtained in a simpler fashion by S. Timoshenko [Schwingungsprobleme der Technik, Springer, Berlin, 1932, p. 344], which neglects the change in the lateral deflection shape with increasing load, and good agreement between the two is found over the entire range investigated. *M. Goland* (Kansas City, Mo.).

Gogoladze, V. G. Reflection and refraction of elastic waves. General theory of boundary Rayleigh waves. Acad. Sci. URSS. Publ. [Trudy] Inst. Séismolog. no. **125**, 43 pp. (1947). (Russian. English summary)

L'auteur étudie la réflexion et la réfraction des ondes élastiques, ainsi que la problème, très important dans la séismologie, des ondes superficielles de Rayleigh. Le premier chapitre contient la discussion du cas des ondes planes: réflexion et réfraction des ondes longitudinales et transversales, distribution d'énergie des ondes hétérogènes, ondes superficielles et équation de Rayleigh. Cette équation est étudiée à fond dans le chapitre 2; on y trouve la discussion complète du comportement des racines de la fonction de Rayleigh sur la surface de Riemann dans des hypothèses variées sur les vitesses. Le mémoire se termine par le critère d'existence des ondes superficielles de Rayleigh et par des formules générales de la réflexion et de la réfraction des ondes élastiques planes. Bibliographie.

V. A. Kostitzin (Paris).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Herzberger, Max. Image error theory for finite aperture and field. I. The image of a point; geometry of the wave surface. J. Opt. Soc. Amer. **38**, 736–738 (1948).

In this first part the author gives a brief but lucid outline of his diapoint theory and states that "the attempts made here consist in approximating a special characteristic function as usual by the section of its Taylor series, but thereafter calculating rigorously, without further approximation, the geometrical qualities of the image formation given by the function chosen." The appearance of the rest of the paper will be awaited with interest by those interested in general aberration-theory.

E. H. Linfoot (Cambridge, England).

Garavaldi, Orestina. Sulle proprietà cardinali dei sistemi ottici. Saggio di una trattazione sintetica. Pont. Acad. Sci. Comment. **6**, 631–653 (1942).

Elementary exposition of the well-known facts of first order aberration (Gaussian) theory in optical systems.

M. Herzberger (Rochester, N. Y.).

Duffieux, P. Michel. Sur un invariant des calculs d'optique géométrique et d'optique physique. C. R. Acad. Sci. Paris **226**, 1257–1259 (1948).

Let $F(x, y)$ represent the complex amplitude-distribution in the exit pupil of an optical system and $G(u, v)$ that in its focal plane. Then, with the usual approximations and normalisations, $F(x, y)$ and $G(u, v)$ are double Fourier transforms of each other. In previous notes [same C. R. **220**, 846–848, 911–913 (1945); these Rev. **7**, 269] the author

has, by nonrigorous mathematical arguments, drawn some incorrect conclusions from this fact, notably that the "moments of illumination"

$$(1)\quad M_u=\iint u^2|G(u,v)|^2dudv,\quad M_v=\iint v^2|G(u,v)|^2dudv$$

are given by the formulae

$$(2)\quad M_u=\iint|\partial F/\partial x|^2dxdy,\qquad M_v=\iint|\partial F/\partial y|^2dxdy,$$

where in (2) the integrals are over the interior of the exit-pupil.

Easy geometrical arguments show that, in fact, the integrals (2) define the moments of ray-density in the geometrical image. In the present note the author gives a proof of this and draws from it the incorrect conclusion that his "moments of illumination" represent an invariant concept common to geometrical and physical optics.

E. H. Linfoot (Cambridge, England).

Svartholm, N. The focal length of a long magnetic lens. Ark. Mat. Astr. Fys. **35A**, no. 6, 9 pp. (1948).

For the computation of the focal length of a magnetic lens an ordinary differential equation of second order has to be integrated. This integration can be carried out approximately if the lens is assumed to be so short that the paraxial rays consist of two sections of straight lines joined together at the lens. One obtains a simple integral expression for the focal length, which was first derived by Busch. The author derives a better approximation by assuming that the magnetic lens has effective values in a finite interval and that the paraxial rays can be represented with sufficient approximation by three sections of straight lines. The resulting formula can be evaluated by numerical integration and implies only a slight increase of numerical work as compared with the evaluation of Busch's formula. The degree of approximation is discussed in a special example.

R. K. Luneburg (New York, N. Y.).

Arzeliès, Henri. Les incidences d'extinction en réflexion vitreuse à deux paramètres. Rev. Optique **27**, 137–156 (1948).

A detailed discussion of plane electromagnetic waves (a) at a plane interface, (b) near a thin plane-parallel plate, the values of K and μ being given for each of the media involved.

E. H. Linfoot (Cambridge, England).

Arzeliès, Henri. Propriétés de l'onde évanescente obtenue par réflexion totale. (Étude théorique). Rev. Optique **27**, 205–244 (1948).

A detailed general study of total reflexion phenomena for plane electromagnetic waves at a plane interface.

E. H. Linfoot (Cambridge, England).

Abelès, Florin. Sur la propagation des ondes électromagnétiques dans les milieux stratifiés. Ann. Physique (12) **3**, 504–520 (1948).

The paper discusses, on the basis of Maxwell's equations, multilayer reflection and refraction of plane electromagnetic waves at plane interfaces, all parallel to each other, in a stratified medium. *E. H. Linfoot* (Cambridge, England).

Toraldo di Francia, Giuliano. Le onde evanescenti nella diffrazione. Ottica **7**, 197–205 (1942).

Toraldo di Francia, Giuliano. Alcuni fenomeni di diffrazione trattati mediante il principio dell'interferenza inversa. Ottica **7**, 117–136 (1942).

Toraldo di Francia, G. Il principio di Huygens-Fresnel come consequenza di quello dell'interferenza inversa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 356–360 (1947).

Levine, Harold, and Schwinger, Julian. On the theory of diffraction by an aperture in an infinite plane screen. I. Physical Rev. (2) **74**, 958–974 (1948).

The authors examine theoretically the diffraction of a scalar plane wave by an aperture in an infinite plane screen, on the assumption that the resultant wave function vanishes at the screen. On the shadow side, the wave function is expressed in terms of its values in the aperture, by means of a surface integral which contains the normal derivative of a Green's function. A similar expression holds on the lit side of the screen, where the incident and reflected primary waves are to be added. The wave field so constructed vanishes automatically at the screen; moreover, it is continuous in passing through the aperture. From the requirement that also the normal derivative be continuous, an integral equation for the aperture field is obtained. Without attempting to solve this equation, the authors utilize it in obtaining the amplitude of the diffracted spherical wave in the far zone in a form that is stationary with respect to small variations of the aperture fields arising from a pair of incident waves. In addition, the transmission cross section of the aperture for a plane wave is shown to be simply proportional to the imaginary part of the amplitude of the transmitted wave in the direction of incidence of the primary wave. The variational principle is applied particularly to the limiting cases of very large and very small wavelengths. Exact results available for the transmission cross section of a circular aperture for normal incidence are compared with the results of the authors' method. It appears that suitable trial aperture fields yield remarkably accurate expressions for the transmission cross section over a wide range of frequencies. In the appendix, the integral

$$\int_0^\infty v^{-n-m}(v^2-1)^{\frac{1}{2}}J_{n+\frac{1}{2}}(\alpha v)J_{m+\frac{1}{2}}(\alpha v)dv$$

is evaluated for $m=1, 2$ and $n=1, 2$. [In the integral equation (2.9) the first k should be ik.] *C. J. Bouwkamp.*

Fel'd, Ya. N. On the principle of duality in the theory of the diffraction of electromagnetic waves by plane screens. Doklady Akad. Nauk SSSR (N.S.) **60**, 1165–1167 (1948). (Russian)

Consider a plane, perfectly conducting screen S with a hole in it; let s denote the part of the screen which is missing. The full plane $S+s$ divides space into the upper half-space and the lower half-space. Let a given source-system in the upper half-space produce the field $\mathbf{E}$, $\mathbf{H}$ in the presence of S and the field $\mathbf{E}^0$, $\mathbf{H}^0$ in its absence. Let a second source-system in the upper half-plane produce the field $\mathcal{E}$, $\mathcal{H}$ in the presence of s and the field $\mathcal{E}^0$, $\mathcal{H}^0$ in its absence. Let the two source systems be so related that, in the lower half-space, $\mathcal{E}^0=\mathbf{H}^0$, $\mathcal{H}^0=-\mathbf{E}^0$. Then in the lower half-space there holds the following analogue of Babinet's principle: $\mathbf{E}=\mathbf{E}^0+\mathcal{H}$, $\mathbf{H}=\mathbf{H}^0-\mathcal{E}$. *E. H. Linfoot.*

Heins, Albert E. The radiation and transmission properties of a pair of parallel plates. II. Quart. Appl. Math. 6, 215–220 (1948).

This paper completes the author's investigation [same Quart. 6, 157–166 (1948); these Rev. **10**, 89] concerning the radiation and transmission properties of a pair of semi-infinite nonstaggered parallel plates, in that a rigorous expression is given for the reflection coefficient of the dominant wave traveling between the plates towards the open end. *C. J. Bouwkamp* (Eindhoven).

Hostinský, B. Modèle mécanique des tensions électrostatiques de Maxwell. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1947**, no. 14, 7 pp. (1948). (Czech. French summary)

Liebmann, G. The electrostatic field distribution near a circular aperture or short cylinder. Philos. Mag. (7) **39**, 281–296 (1948).

The author is concerned with the computation of the electrostatic potential near a circular aperture in a conducting sheet of finite thickness immersed in a field that, if the opening were absent, would be homogeneous on either side of the sheet and normal to it. For an infinitely thin sheet the problem has been solved rigorously [cf. F. Ollendorff, Potentialfelder der Elektrotechnik, Springer, Berlin, 1932, pp. 295–298]. This rigorous solution, in combination with Bertram's method for calculating fields within a cylinder when the potential distribution along its surface is known [J. Appl. Physics **13**, 496–502 (1942); these Rev. **4**, 91] suffices to solve the problem in question approximately. Various diagrams and tables are presented.

C. J. Bouwkamp (Eindhoven).

Gauster-Filek, W. Wechselfelder, Kreisdrehfelder und elliptische Drehfelder. Österreich. Ing.-Arch. **1**, 394–407 (1947).

This paper is a general discussion of methods of representing alternating and circular or elliptically rotating fields by means of uniformly rotating and harmonically pulsating vectors and their combinations. An application to 3-phase electrical systems is suggested. *L. C. Hutchinson.*

Ceĭtlin, L. A. The capacity of curvilinear conductors. Doklady Akad. Nauk SSSR (N.S.) **59**, 1583–1586 (1948). (Russian)

The determination of the capacity of a system of conductors reduces to the determination of the coefficients of potential p_{rs} [see, for example, Jeans, The Mathematical Theory of Electricity and Magnetism, 5th ed., Cambridge, 1933, chap. IV]. Under the simplifying assumption that the corresponding charge is uniformly distributed on each conductor, the author, in a paper unavailable to the reviewer [Trudy Voen. Èlektrotehn. Akad. **1944**, no. 7], claims to have obtained a "sufficiently accurate solution" to the problem of determining p_{rs} for a system of straight uniform conductors. It is said that the determination of p_{rs} $(r\neq s)$ when the conductors are curvilinear can be made to depend on this solved case so that the present paper is concerned with p_{rr} which, under the given assumption, is $\epsilon^{-1}A^{-2}\int\int D_{12}^{-1}dA_1dA_2$, where D_{12} is the distance between the elements dA_1 and dA_2 of the surface S of area A over which the integration is extended and ϵ is the dielectric constant.

By assuming that the curvilinear conductor has a uniform cross-section and that its axis is a smooth (plane for convenience) curve that is not too sharply bent and does not come too close to itself, relative to the maximum diameter of the cross-section, it is found that p_{rr} can be closely approximated by the sum of two terms one of which depends only on the form and size of the cross-section while the other depends only on the form and size of the axis of the conductor. In the special case of a circular ring with circular cross-section, there is good agreement with the previously known result. *R. Church* (Annapolis, Md.).

Hössjer, Gustav. On the foundations of electrodynamics. Trans. Chalmers. Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar] no. **69**, 13 pp. (1948).

It is shown that Maxwell's equations can be obtained from the surface integrals

$$I_1=\int\int[A_xdydz+A_ydzdx+A_zdxdy+B_xdldx+B_ydldy+B_zdldz],$$
$$I_2=\int\int[B_xdydz+B_ydzdx+B_zdxdy+A_xdldx+A_ydldy+A_zdldz]$$

by imposing the integrability condition and by identifying the vectors **A** and **B** with the electromagnetic field vectors i**E** and **H**, respectively. The relation of the present method of developing electrodynamics to the theory of complex variables is pointed out. *C. Kikuchi.*

Belinfante, F. J. A first-order variational principle for classical electrodynamics. Physical Rev. (2) **74**, 779–781 (1948).

In the usual formulation of classical mechanics, the variational principle applied to the appropriate Lagrangian leads to equations of motion of the second order in the time variable t, whereas in quantum mechanics equations of the first order in t are obtained. The present note seeks to remove this difference by using Lagrangians which will lead to equations of the first order in t. The case for a classical point electron is treated specifically, and the resemblance of the resulting Hamiltonian to that of Dirac's theory is pointed out. *C. Kikuchi* (East Lansing, Mich.).

Feynman, R. P. A relativistic cut-off for classical electrodynamics. Physical Rev. (2) **74**, 939–946 (1948).

The present paper investigates the effect of relaxing the condition that interaction between charged particles occur only when the four-dimensional interval $S^2=t^2-r^2$ is exactly zero. It is assumed that substantial interaction exists as long as the interval S is time-like and less than some small constant a. It is shown that this modification does not produce any appreciable effect on the interaction of charges which are several electron radii apart. The possibility of pair production and annihilation in such a classical field is pointed out. The theory developed here is similar to those of F. Bopp [Ann. Physik (5) **42**, 573–608 (1942); these Rev. **8**, 124] and B. Podolsky and P. Schwed [Rev. Modern Physics **20**, 40–50 (1948); these Rev. **9**, 551].

C. Kikuchi (East Lansing, Mich.).

Ashauer, Sonja. A generalization of the method of separating longitudinal and transverse waves in electrodynamics. Proc. Roy. Soc. London. Ser. A. **194**, 206–217 (1948).

A generalization of the method of separating longitudinal and transverse waves in electrodynamics is proposed. It

consists in splitting up each Fourier-component of the wave-field 4-vector with respect to two null vectors $\mathbf{k}$ and $\mathbf{l_K}$ where $\mathbf{k}$ is the Fourier vector of propagation and $\mathbf{l_K}$ is an arbitrary (real) function of $\mathbf{k}$ satisfying $(\mathbf{l_K}, \mathbf{k})=1$, $\mathbf{L_K}=-\mathbf{l_K}$. An application to the motion of an electron in the field of the nucleus is considered. *From the author's summary.*

Supek, I. Dispersion de la lumière pour les ondes de Dirac sans quantisation. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **3**, 17–22 (1948). (Croatian. French summary)

Carrara, N. Deduzioni relativistiche sulla propagazione del campo elettromagnetico entro una guida d'onda. Nuovo Cimento (9) **5**, 249–262 (1948).

Abele, M. Teoria della propagazione di un campo elettromagnetico lungo una guida dielettrica a sezione circolare. Nuovo Cimento (9) **5**, 274–284 (1948).

Taub, A. H. Orbits of charged particles in constant fields. Physical Rev. (2) **73**, 786–798 (1948).

The author denotes the four-dimensional velocity vector by $V^\mu = dx^\mu/ds$ and discusses the solutions of the equations $d^2x^\mu/ds^2 = \lambda f_\nu{}^\mu dx^\nu/ds$, where

$$\|f_\nu{}^\mu\| = \left\| \begin{matrix} 0 & H_3 & -H_2 & E_1 \\ -H_3 & 0 & H_1 & E_2 \\ H_2 & -H_1 & 0 & E_3 \\ E_1 & E_2 & E_3 & 0 \end{matrix} \right\| = \mathfrak{F}$$

and $\lambda = e/(m_0c^2)$. Here E_i and H_i are the components of the electric and magnetic field strengths, respectively. These equations may be written in matrix form as $dV/ds = \lambda\mathfrak{F}V$, where $V = \|V^\mu\|$ and is a one-column matrix. For V we get $V = L(s)V_0 = e^{\lambda s\mathfrak{F}}V_0 = (1+\lambda s\mathfrak{F}+\lambda^2s^2\mathfrak{F}^2/2!+\cdots)V_0$, if the tensor $f_\nu{}^\mu$ is constant. Because $L(s) = e^{\lambda s\mathfrak{F}}$, $L(s)$ is a proper Lorentz matrix. When $L(s)$ is determined, the orbit may be obtained by a quadrature. The particle undergoes constant acceleration b^* in the sense of special relativity, of the magnitude $b^{*\sigma}b_\sigma{}^* = \lambda^2 f_{\lambda\beta}f_\gamma{}^\beta V_0{}^\lambda V_0{}^\gamma$. Furthermore the author determines the matrix $L(s)$ in terms of $\mathfrak{F}$, $\mathfrak{F}^2$, $\mathfrak{F}^3$ and $\mathfrak{F}^4$, the coefficients of these matrices being functions of s and certain scalars determined by the tensor $f_\nu{}^\mu$. In the parabolic case both terms vanish: $a = \sum_{i=1}^3(E_i^2-H_i^2)=0$, $b=\sum_{i=1}^3 E_iH_i=0$. In this case we get $L(s)=1+\lambda s\mathfrak{F}+\lambda^2s^2\mathfrak{F}^2/2!$ and the integral $x = x_0+sV_0+\frac{1}{2}\lambda s^2FV_0+\frac{1}{6}\lambda^2s^3F^2V_0$, where x_0 is the one-column matrix $x_0=\|x_0{}^\mu\|$ and $x_0{}^\mu$ are the constant space-time coordinates of the position of the particle when $s=0$. For the further determination of the particle orbit the author uses a coordinate system for which

$$\|f_\nu{}^\sigma\| = H \left\| \begin{matrix} 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{matrix} \right\|$$

is valid. In this case there are no spatially "closed" orbits. In case H is a function of the space-time coordinates the equations (*) describe the electromagnetic field of a plane wave progressing in the x^1-direction.

Applying these methods the author obtains a system of parametric equations for the orbit. The nature of these orbits is discussed in detail for the special case where $H(s) = \sin 2\pi fs$. In the nonparabolic case the author states the general parametric equations of the orbit, of which he discusses special cases. Furthermore, a necessary and sufficient condition is stated that the orbit is periodic in the sense that there exists a constant σ so that $V^\mu(s+\sigma) = V^\mu(s)$. There is also an examination in detail of the proper values of Lorentz matrices in the parabolic case and in the nonparabolic case. *M. Pinl* (Cologne).

Aymerich, Giuseppe. Sul moto prossimo a quello piano di un corpuscolo elettrizzato in un particolare campo magnetico di tipo notevole. Rend. Sem. Fac. Sci. Univ. Cagliari **16** (1946), 149–164 (1948).

Størmer, Carl. Résultats des calculs numériques des trajectoires des corpuscules électriques dans le champ d'un aimant élémentaire. VI. Trajectoires par l'origine. Faisceau fondamental. Skr. Norske Vid. Akad. Oslo. I. **1947**, no. 1, 81 pp. (3 plates) (1947).

Part V appeared in the same Skr. **1936**, no. 6.

Cagniard, L. Sur les phénomènes d'induction électromagnétique dans une sphère conductrice. Applications géophysiques. Ann. Fac. Sci. Univ. Toulouse (4) **8** (1944), 59–88 (1947).

Lamb studied the hypothesis that the earth's magnetic field with its secular variation can be explained by extremely slowly decaying electrical currents induced at a certain moment in the past in the globe's metallic nucleus when an exterior magnetic field suddenly vanished. The author proposes and develops a new solution of the mathematical problem of the gradual decay of these induced currents. His final formulas present an advantage when compared with Lamb's solution: they are so simple that the vanishing of the induced currents can be studied numerically in detail. Among many results one important fact is proved: the exterior magnetic field of decaying electrical currents must be identical with that of an infinitesimal magnet located at the earth's center; this is precisely the fundamental feature of the earth's magnetic field.

The author does not take into consideration the rotation of the earth, and his computations (like those of Lamb) apply only to a stationary globe. He expresses the hope that a more complete theory of the phenomenon, which would take into consideration the rotation of the earth, could perhaps explain also the motion of earth's magnetic poles and the secular variation of its magnetic field as well as the natural electrical currents circulating in the oceans and continents. *E. Kogbetliantz* (New York, N. Y.).

Chapman, Sydney. The supposed fundamental geomagnetic field. Ann. Géophysique **4**, 109–123 (1948).

This paper discusses the hypothetical (so called "fundamental") magnetic field of rotating bodies, whose supposed existence is based on the assumption (proposed by H. A. Wilson) that a moving element of mass m grams creates a magnetic effect as if it were a moving charge of $mG^{\frac{1}{2}}$ esu., where G denotes the gravitational constant.

The earth's fundamental magnetic field is calculated outside and inside the globe, assuming perfect spherical symmetry. For the exterior points there is no difference between the fundamental field and the field of the classical theory, according to which the geomagnetic field originates below the nonmagnetic rocky crust of the earth. But, as the author proves, for the interior of the globe, the behavior of the horizontal component is opposite for these two fields: whereas in the core-theory the horizontal component must increase with the depth, the fundamental field is characterized by a decrease of horizontal component with a reversal of its sign at a depth (circa one tenth of radius) depending on the density distribution.

The author points out that the observations of the magnetic intensity in South Africa, to a depth of a mile, indicate a decrease of intensity with depth of the right order of magnitude. The theory states that this decrease is very small: 0.3% per mile of depth, if the vertical component is negligible (the vertical component increases with depth in both theories). In the reviewer's opinion, the local magnetic anomalies whose order of magnitude is the same may create an incorrect impression that the geomagnetic field decreases, since the increase of horizontal component in the core-theory is extremely small: only 0.075% per mile of depth.

E. Kogbetliantz (New York, N. Y.).

Quantum Mechanics

***Mott, N. F., and Sneddon, I. N. Wave Mechanics and its Applications.** Oxford, at the Clarendon Press, 1948. xii+393 pp. $10.00.

This book is intended for readers who are interested in the applications of quantum mechanics rather than in its logical foundations or general theoretical development. The wave equation is introduced as quickly as possible, on the basis of a brief discussion of the relevant experimental facts, and the wave-mechanical method is used throughout, the ideas of matrix mechanics and of transformation theory being discussed only briefly in the last chapter. Roughly half of the book is devoted to simple illustrations of wave mechanics and the application to the electronic structure of atoms; in this part are included discussions of the WKB (phase-integral) method, perturbation and variation methods, electron spin, and the Hartree and Fock methods. The remaining half contains material on interatomic forces and valence, the theory of solids, collision problems, radiation theory and relativistic quantum theory. The variety of information on physical problems included in these chapters is even more remarkable than that in the earlier chapters.

On some points the explanations are so brief as to lack clarity, and, as in many publications of the last twenty years, the number of misprints and minor errors is large enough to be troublesome. In all important matters, however, the treatment appears sound and correct, and the physical viewpoints used are interesting and stimulating. The wide and varied scope of subjects discussed in a single volume will make the book most useful as an introduction to fields in which active research is progressing. References to periodical literature and to treatises are provided.

W. H. Furry (Cambridge, Mass.).

Feynman, R. P. Space-time approach to non-relativistic quantum mechanics. Rev. Modern Physics **20**, 367–387 (1948).

A new formulation of quantum mechanics is developed which is mathematically equivalent to the Schrödinger and Heisenberg formulations. Probability amplitudes are assigned to the entire motion of a particle in time instead of to its position at a particular time. The probability that a particle will be found by measurement to have a path in a certain region of space-time is taken to be the absolute square of the sum of complex contributions from all paths of the region. The different paths contribute equally in magnitude to this probability, but each with a phase equal, in units of $h/2\pi i$, to the classical action, that is, the time integral of the Lagrangian along the path. This summing of the contributions from different paths amounts mathematically to the integration of a functional over a region of Hilbert space. The author outlines in particular cases the limiting processes involved in such an integration. That this theory is equivalent to Schrödinger's is shown by deriving the wave equation in it. A form of Huyghens' principle is shown to hold for matter waves. The relation of the theory to the Heisenberg matrix mechanics is also exhibited. Some applications are given, including the elimination, from the equations of quantum electrodynamics, of the coordinates of the oscillators used to represent the field.

O. Frink (State College, Pa.).

Bass, Jean. Application aux mélanges de la théorie du transfert des grandeurs aléatoires. C. R. Acad. Sci. Paris **226**, 1351–1353 (1948).

This paper is a brief and, in many respects, incomplete outline of a connection between the density matrix of certain quantum statistical systems, and stochastic processes. Certain formal operators are introduced which serve for the definition of the characteristic function for the latter, as well as for its complex generalization of a distribution function. A hydrodynamical analogue is mentioned, in which a "mixture" corresponds to a rotational flow, a "pure state" to an irrotational one. Insufficient detail is given to permit of further comment.

B. O. Koopman.

Bass, Jean. Sur les moyennes et les lois de probabilité en mécanique ondulatoire. C. R. Acad. Sci. Paris **227**, 112–114 (1948).

Morette, Cécile. Quelques propriétés des ensembles de mouvements possibles en mécanique ondulatoire en vue d'une mécanique ondulatoire statistique. Cahiers de Physique nos. **31–32**, 63–74 (1948).

The author extends the theory of "previsions" of J. L. Destouches in wave mechanics to a type of quantum statistics. Rather than a single initial wave function ψ_0 from which $\psi(t)$ is to be predicted, she assumes a discrete or continuous aggregate of such initial wave functions, with a prescribed law of distribution of these initial states, from which a statistical aggregate of trajectories in Hilbert space is to be determined. She examines some particular cases, and considers the resulting probability laws for quantities such as the energy connected with the system.

O. Frink (State College, Pa.).

Landsberg, P. T. An algebra of observables. Philos. Mag. (7) **38**, 757–773 (1947).

A quantum-mechanical formalism is proposed dealing with quantities called observables, variables and operators. In the notations used, $[A]$ and $[B]$ represent observables, A and B the corresponding Hermitian operators, while $[A]+[B]$ and $c[A]$ are examples of variables. Variables are quantities computed from observables. Observables represent quantities directly observed, and are always denoted by expressions with overall brackets, such as $[[AB]B]$. Observables with internal brackets, as in this example, are called pseudo-observables; their physical meaning requires interpretation. An observable without internal brackets is said to be structureless. The first fundamental assumption is that the product $[AB]$ corresponds to the operator $\frac{1}{2}(AB+BA)$ if $[A]$ and $[B]$ correspond to the operators A and B. Observables constitute essentially a Jordan algebra, multiplication being commutative but not associative. The second assumption is that all observables are expressible in

terms of a basic set $[A_j]$ whose corresponding operators obey commutation rules of the form $i(A_jA_k-A_kA_j)=h_{jk}I$. It is then shown that all pseudo-observables may be expressed in a particular way in terms of structureless observables. The resulting formulas may be interpreted as uncertainty relations. *O. Frink* (State College, Pa.).

Landsberg, P. T. On the occurrence of detailed balancing in quantum mechanics. Philos. Mag. (7) **38**, 824–828 (1947).

The author assumes that observables are represented by spectral sets $\{A_j\}$ of independent idempotent elements of a linear associative algebra over the complex field; also that a unitary element U of this algebra exists transforming a given spectral set $\{A_n\}$ into another $\{B_n\}$ so that $B_n=UA_nU^{-1}$. If A and B are idempotent elements of spectral sets, he deduces the existence of a real transition probability p_{AB} with $0\leqq p_{AB}\leqq 1$, such that $ABA=p_{AB}A$, $BAB=p_{BA}B$, and $p_{AB}=p_{BA}$. This last equality is a formal statement of the principle of detailed balancing, which is thus shown to follow from the other properties of transition probabilities. *O. Frink* (State College, Pa.).

McWeeny, R., and Coulson, C. A. Quantum mechanics of the anharmonic oscillator. Proc. Cambridge Philos. Soc. **44**, 413–422 (1948).

The energy levels and transition probabilities for an oscillator with potential energy a polynomial in the displacement can be found by first expressing the Hamiltonian operator as a matrix in a representation in which the energy of a harmonic oscillator is diagonal, and then bringing this matrix to approximately diagonal form by numerical computation. This method is applied in detail to the case of potential energy proportional to the fourth power of displacement. The frequency of the harmonic oscillator whose states are used as a basis can be chosen arbitrarily, and this choice is made in such a way as to secure rapid convergence. The five lowest levels are computed to a fractional accuracy of about 10^{-6} by using eight rows and columns of the secular matrix. The transformation matrix and the matrix elements determining transitions are also computed. A simple analytic approximation is suggested for the wave function of the lowest state. *W. H. Furry* (Cambridge, Mass.).

Corben, H. C. The use of phase space in classical and quantum theory. Physical Rev. (2) **74**, 788–794 (1948).

The author discusses the classical kinematics of a single charged particle, introducing an 8-dimensional phase space with a metric tensor which is restricted to have 16 independent components representing the ten gravitational potentials and the six electromagnetic field strengths. The formalism developed here is also interpreted in terms of a complex four-dimensional space with a Hermitian metric. Poisson brackets are defined and their interpretation is attempted. A fundamental constant is introduced into the theory and it is expressed in terms of h and a length l. It is shown that as $l\to 0$ the theory here proposed goes over to the usual one. For finite l the Poisson brackets of two coordinates of position do not commute. *A. H. Taub.*

Markov, M. A. A classical analogue of the quantum theory of perturbation. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 510–514 (1948). (Russian)

The Hamilton-Jacobi equation of a classical relativistic system containing charged particles is solved in successive approximations by assuming Hamilton's principal function S to be expressible as a series in ascending powers of the charge e. This procedure is analogous to the perturbation method generally used in the quantum theory of radiation. The method is applied to two problems: the transverse or "magnetic" self-energy of an electron, and the interaction of two fast electrons. The result obtained for the self-energy corresponds to that obtained by Waller from the older form of Dirac theory; in this case quantum methods, in the positron theory, have already provided a more desirable (less divergent) result. The interaction formula is not completely worked out, but is stated to correspond to that of Möller, as would be expected. *W. H. Furry.*

Dirac, P. A. M. Quantum theory of localizable dynamical systems. Physical Rev. (2) **73**, 1092–1103 (1948).

The author calls a dynamical system in quantum theory localizable if its wave function can be expressed in terms of variables, each referring to physical conditions at only one point in space-time. Such a representation will be called localized. These variables may be at points on any three-dimensional space-like surface S. All relativistic atomic models used at present are localizable, but they all lead to difficulties. Are these difficulties directly caused by the models' being localizable? Should we endeavor to construct new theories in terms of quantities that are not accurately localizable? The author answers these questions in the negative and investigates localizable systems along the most general lines, working only with relativistic ideas. He provides a test of whether any theory of particles in interaction is relativistic and gives conditions which must be satisfied for new theories. He makes a general investigation into how the wave function varies when the surface S is varied subject to the condition that it remains space-like and supposes that a state of a dynamical system can be fixed by a wave function $\psi(q)$ involving variables q which are all localized on S. The variables q may consist of discrete variables, denoting positions of particles on S (and their spins) and may also consist of three-fold infinities of variables, denoting field quantities on S. As the surface S plays the role of the time in the nonrelativistic theory, we have many time variables in the present theory, namely all those needed to fix S.

For two surfaces S, say S_1 and S_2, and two wave functions $\psi_1(q_1)$ and $\psi_2(q_2)$, we have to use different variables q_1 and q_2. The two functions are connected by an equation of the form (*) $\psi_2(q_2)=R\psi_1(q_1)$, where R is a unitary operator. In order to study the consequences of the linear connection (*) between wave functions on different surfaces the author assumes S_1 and S_2 to differ only by an infinitesimal of order ϵ. Equation (*) now gives

$$(**)\qquad \psi_2(q_2)-\psi_1(q_1)=-i\epsilon A\psi_1(q_1),\quad R=1-i\epsilon A,$$

where A is a linear operator. The condition that R is unitary requires A to be Hermitian.

A parametrized surface S_1 is a surface with a system of curvilinear coordinates u_1, u_2, u_3 on it; it is fixed by specifying the four coordinates x_μ in space-time ($\mu=0, 1, 2, 3$) of any point on the surface as function of the parameters u_r ($r=1, 2, 3$), i.e., it is fixed by the four functions $x_\mu=x_\mu(u)$. The author investigates a small deformation of S_1, which is fixed by specifying the variation δx_μ in the coordinates of a point with any given parameters u, $\delta x_\mu=\delta x_\mu(u)$. He resolves the displacement δx_μ into a part lying in the surface S_1 and a part normal to the surface S_1:

$$\delta x_\mu=(\partial x_\mu/\partial u_r)\epsilon a_r+n_\mu\epsilon a_n$$
$$(n_\mu(\partial x^\mu/\partial u_r)=0,\ n_\mu n^\mu=1,\ n_0>0)$$

and determines two advantages: first, one may apply small deformations with a_r, a_n any differentiable functions of the u's to a space-like parametrized surface S_1 and it always remains space-like; second, the ϵa_r, ϵa_n specification allows one immediately to separate out those deformations for which only the coordinate system u is changed, corresponding to a trivial change in the wave function. The general variation of the wave function is given by equations of the Schrödinger type involving certain operators $P^n(u)$ which play the role of Hamiltonians. While the nonrelativistic quantum theory has just one Schrödinger equation, the present theory has a large number, one for each kind of deformation that can be applied to a parametrized surface S. The author compares the present theory with the many-time theory of electrons in interaction with the electromagnetic field, which has many wave equations, one for each electron. If $F(S_1)$ is any function of S_1, i.e., any functional of the functions $x_p(u)$ that specify S_1, the same function $F(S_2)$ of the deformed S_2 will be given by $F(S_2)-F(S_1)=-i\epsilon\mathfrak{A}F(S_1)$ (to the first order in ϵ) in which $\mathfrak{A}$ is the deformation operator. With the help of this notation equations (**) become $\mathfrak{A}\psi_1(q)=A\psi_1(q)$ or, if we let $\psi(qS)$ be the wave function on an arbitrary S, (***) $\mathfrak{A}\psi(qS)=A\psi(qS)$ (of the Schrödinger type). For the operators $\mathfrak{A}$ and A, the author gets

$$\mathfrak{A}=\int\{a_r(u)\Pi^r(u)+a_n(u)\Pi^n(u)\}du_1du_2du_3,$$

$$A=\int\{a_r(u)P^r(u)+a_n(u)P^n(u)\}du_1du_2du_3.$$

The fundamental commutation relations for the Hamiltonians $P^n(u)$ are obtained, of the form

$$[P^r(u),P^n(u')]+[\Pi^n(u'),P^r(u)]-[\Pi^r(u),P^n(u')]=\frac{\partial\delta(u-u')}{\partial u_r}P^n(u),$$

$$[P^n(u),P^n(u')]+[\Pi^n(u'),P^n(u)]-[\Pi^n(u),P^n(u')]=\frac{\partial\delta(u-u')}{\partial u_r}\{\gamma_{rs}(u)P^r(u)+\gamma_{rs}(u')P^r(u')\},$$

$$\gamma^{rs}=-(\partial x_p/\partial u_r)(\partial x_p/\partial u_s).$$

The author states that any relativistic quantum theory of a localizable dynamical system will provide an example of linear operators $P^n(u)$ satisfying these commutation relations. Conversely, any example of linear operators $P^n(u)$ satisfying these relations will provide a relativistic theory of a localizable dynamical system. [Cf. W. Heisenberg, Z. Physik **120**, 513–538, 673–702 (1943); these Rev. **4**, 292; W. Heisenberg and W. Pauli, Z. Physik **56**, 1–61 (1929); P. Weiss, Proc. Roy. Soc. London. Ser. A. **169**, 102–133 (1938); S. Tomonaga, paper reviewed below.]

M. Pinl (Cologne).

Tomonaga, S. On a relativistically invariant formulation of the quantum theory of wave fields. Progress Theoret. Physics **1**, 27–42 (1946).

A formalism is presented for writing the quantum theory of wave fields so that the Lorentz invariance of the theory is everywhere apparent. No modification of the theory is introduced, so that the divergence difficulties remain, but the development is no longer on lines which parallel those of nonrelativistic theory. The theory is developed for the case of two interacting fields under the assumptions that the interaction energy density is a scalar and that the energy densities at different points at the same time commute. The well-known commutation relations between the field variables and their canonically conjugate variables are first thrown into a relativistic form by means of a unitary transformation. The ψ vector obtained by this transformation is then seen to satisfy an equation which is analogous to that introduced in Dirac's many time formalism [Proc. Roy. Soc. London. Ser. A. **136**, 453–464 (1932)] if one introduces here infinitely many time variables t_{xyz}, one for each point in space. The ψ vector is then a functional of t_{xyz}, and the wave equations become an infinite set of functional partial differential equations which may be expressed in a manner which is independent of any reference frame:

$$\left\{H_{12}(P)+\frac{h}{i}\frac{\delta}{\delta C_P}\right\}\psi[c]=0,$$

where $\psi[C]$ is a functional of the hypersurface C and $H_{12}(P)$ the interaction energy density between the two fields at point P on C. A generalized probability amplitude is defined to give a physical meaning to the ψ functional.

H. C. Corben (Pittsburgh, Pa.).

Koba, Zirô, Tati, Takao, and Tomonaga, Sin-itiro. On a reletivistically invariant formulation of the quantum theory of wave fields. II. Case of interacting electromagnetic and electron fields. Progress Theoret. Physics **2**, 101–116 (1947).

The Tomonaga formalism [cf. the preceding review] is applied to the case of interacting electromagnetic and electron fields. After applying a unitary transformation to throw the commutation relations into an obviously Lorentz invariant form, the interaction energy density $H_{12}(P)$ of Tomonaga's equation is defined. It is shown that the interaction energy density at two points lying outside each other's light cones commute, even for points infinitesimally close. This condition implies that Tomonaga's equation is integrable when the independent variable surface C is space-like. An explicit expression is found for $f_X[C]$, a functional of C and a function of position X, such that $\Xi_X[C]=\operatorname{div}A(X)+f_X[C]$ commutes with the Hamiltonian functional, so that the auxiliary condition may be consistently imposed. Thus only linear operators which commute with $\Xi_P[C]$ (P on C) represent physically measurable quantities, and it is shown that these are gauge invariant quantities.

H. C. Corben (Pittsburgh, Pa.).

Koba, Zirô, Ôisi, Yasuharu, and Sasaki, Muneo. Auxiliary condition and gauge transformation in the "super-many-time theory." I. Progress Theoret. Physics **3**, 141–151 (1948).

The auxiliary condition introduced earlier [see the preceding review] is derived alternatively by use of the Heisenberg picture and is applied successively to interacting electron-photon, photon-scalar meson, photon-vector meson and nucleon-vector meson fields.

H. C. Corben.

Koba, Zirô, Tati, Takao, and Tomonaga, Sin-ichirô. On a relativistically invariant formulation of the quantum theory of wave fields. III. Case of interacting electromagnetic and electron fields. Progress Theoret. Physics **2**, 198–208 (1947).

The integrability condition $(H_{12}(P), H_{12}(P'))=0$ (P, P' outside each other's light cones) implies that Tomonaga's

equation [cf. the third preceding review] may be integrated in terms of the unitary operator

$$T[C, C_0]=\prod_{C_0}^{C}(1-(i/\hbar)H_{12}(x)dV_x),$$

which may be regarded as a transformation functional $\psi[C]=T[C, C_0]\psi[C_0]$. In virtue of this integrability for the interaction of electron and electromagnetic fields it is shown that the usual formalism of quantum electrodynamics may be derived from the Tomonaga formalism as a special case in which the surface C in Tomonaga's equation is a plane parallel to the (x, y, z)-plane. It is pointed out that in the case of a meson field interacting with nucleons or with the electromagnetic field the integrability condition no longer holds. *H. C. Corben* (Pittsburgh, Pa.).

Kanesawa, Suteo, and Tomonaga, Sin-itiro. On a relativistically invariant formulation of the quantum theory of wave fields. IV. Case of interacting electromagnetic and meson fields. Progress Theoret. Physics **3**, 1–13 (1948).

[In the original the paper was incorrectly numbered V.] The relativistic formulation of quantum electrodynamics given earlier [cf. the third preceding review] is extended to the theory of interacting electromagnetic and scalar and pseudoscalar meson fields. Here the interaction energy of the fields is not a scalar and the energy densities at two infinitesimally space-like separated world points do not commute. It is therefore necessary to add to the interaction term in the Hamiltonian a functional such that the sum is a scalar and the sum when substituted into Tomonaga's generalized Schrödinger equation allows one to throw the latter into an integrable form. It is shown that it is possible to choose such a functional. An auxiliary condition analogous to that introduced into quantum electrodynamics is defined and shown to be compatible with the equations of motion. *H. C. Corben* (Pittsburgh, Pa.).

Kanesawa, Suteo, and Tomonaga, Sin-itiro. On a relativistically invariant formulation of the quantum theory of wave fields. V. Case of interacting electromagnetic and meson fields. Progress Theoret. Physics **3**, 101–113 (1948).

The theory developed in a previous paper [cf. the preceding review] for the interaction of electromagnetic and scalar or pseudoscalar meson fields is developed here in analogous form for the interaction of electromagnetic and vector or pseudovector meson fields. As before the interaction energy is not a scalar and the energy densities at two adjacent points do not commute, but it is possible to define a term to be added to the Hamiltonian which makes Tomonaga's equation integrable. *H. C. Corben.*

Tanikawa, Yasutaka. On the generalized transformation functions. Progress Theoret. Physics **1**, 12–20 (1946).

If we have a complete set of commuting observables $q\equiv q_i$, the square of the modulus of the Dirac transformation function $\psi(q_T''|q_t')$ gives the relative a priori probability of any state yielding the results q_t' and q_T'' when the q_i are observed at times t, T. Then $\psi(q_T''|q_t')=(q_T''|D(T-t)|q_t')$. Denote by Mq' the operator corresponding to a process of observation which certainly gives the results q'. The generalized transformation function considered here may then be written $T(q_T'', q_t')=Mq_T''D(T-t)Mq_t'$. When this is applied to the Tomonaga formalism of the quantum theory of fields [cf. the sixth preceding review] the generalized transformation function may be defined as a functional of a closed three-dimensional region of space-time. The composition law for such functionals of two subregions with common boundary surface is satisfied only when this boundary extends in space directions and lies entirely outside a light cone having its boundary on the surface and when the total region is completely separated into two subregions by any space-like surface which includes the common surface and extends to infinity. *H. C. Corben* (Pittsburgh, Pa.).

Tanikawa, Y. The supplementary note to the generalized transformation function. Progress Theoret. Physics **2**, 219–220 (1947).

By introducing functions of noncommuting variables Dirac [Rev. Modern Physics **17**, 195–199 (1945); these Rev. **7**, 180] has extended the analogy between classical and quantum contact transformations, and in this letter it is pointed out that in terms of the generalized transformation functions introduced by the author such functions may in fact be defined. [Cf. the preceding review.]

H. C. Corben (Pittsburgh, Pa.).

Watanabe, Satosi. On Dirac's general transformation function. I. Progress Theoret. Physics **2**, 71–88 (1947).

After discussing the fundamental notions of functional analysis and applying them to curvilinear coordinates in Minkowski space, the author postulates a variational principle for a number of classical fields u^a:

$$\delta\int \mathfrak{L}(u^a, \partial u^a/\partial x^\mu, x^\mu)dx^0dx^1dx^2dx^3=0,$$

where the integral is regarded as a functional of the u^a. Before passing to the analogy with the formalism of mass points, a transformation to curvilinear coordinates $\xi^\mu(\mu=0, 1, 2, 3)$ is introduced, and one of these, $s=\xi^0$, is assigned a role similar to that of time in ordinary mechanics. The Lagrangian and Hamiltonian formalisms are then developed in terms of a Lagrangian functional L obtained by integrating $\mathfrak{L}\partial(x)/\partial(\xi)$ over the other three ξ^μ, and in terms of a corresponding Hamiltonian functional. The theory of canonical transformations and their generating functionals is then developed by analogy with the standard formulation. The similarity of the generating functional $\int Lds$ and Dirac's general transformation function is demonstrated.

H. C. Corben (Pittsburgh, Pa.).

Utiyama, Ryôyû. On the canonical transformation in quantum theory. Progress Theoret. Physics **2**, 117–126 (1947).

The well-known transformation theory for a system with one degree of freedom is summarized for the classical treatment and for the Heisenberg and Schrödinger pictures of the quantum mechanical treatment. The analogous formalism is then developed for the quantum theory of two interacting fields in terms of the Tomonaga method, leading to a functional differential equation which corresponds to the Hamilton-Jacobi equation of classical mechanics.

H. C. Corben (Pittsburgh, Pa.).

Miyamoto, Yonezi. On the interaction of the meson and nucleon field in the super-many-time theory. Progress Theoret. Physics **3**, 124–140 (1948).

Tomonaga's formalism is developed for the case of interacting meson and nucleon fields, where as in the meson-photon case the interaction energy density is not a scalar.

The generalized Schrödinger equations are given for the cases of scalar and vector mesons, with details of the calculation for the latter. A summary of Stückelberg's formulation [Helvetica Phys. Acta 11, 299–328 (1938)] for the vector meson field is given and the introduction of the auxiliary condition discussed. The transformation to the usual formalism is demonstrated when the variable surface is taken normal to the time axis. *H. C. Corben.*

Kanai, Eizo. Some remarks on the non-infinitesimal commutation relations. Progress Theoret. Physics **2**, 135–144 (1947).

It has been thought that some difficulties in the quantum theory of fields may be due to δ-function commutation relations. Here commutation relations are sought for a complex scalar field compatible with $(\Box-\kappa^2)\psi=0$ and invariant for Lorentz transformations. A possible solution avoiding δ-functions is

$$[\psi(\mathbf{x},t),\psi^*(\mathbf{x}',t)]_\pm=f_1(\mathbf{x}-\mathbf{x}'),\ [\Pi(\mathbf{x},t),\Pi^*(\mathbf{x}',t)]_\pm=f_2(\mathbf{x}-\mathbf{x}')$$

with $f_2(\mathbf{x})=-(\nabla^2-\kappa^2)f_1(\mathbf{x})$, $f_1=-(g\pi\kappa/2\sqrt{R})H_1^{(1)}(i\kappa\sqrt{R})$, Π being the momentum conjugate to ψ and $R=x^2+y^2+z^2$. The Hamiltonian is

$$ihc\int\rho(\mathbf{x}',\mathbf{x}'')[\psi^*(\mathbf{x}')\Pi(\mathbf{x}'')-\Pi^*(\mathbf{x}'')\psi(\mathbf{x}')]d\mathbf{x}'d\mathbf{x}'',$$

where $\int f_1(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}'-\mathbf{x}'')d\mathbf{x}'=\delta(\mathbf{x}-\mathbf{x}'')$, and is Hermitian but not positive definite. To avoid negative energy states by the use of hole theory the quantisation must be in accordance with Fermi-Dirac statistics but this contradicts the relation between spin and statistics. *C. Strachan.*

Kanai, Eizo, and Takagi, Shuji. Some remarks on Bopp's field theory. Progress Theoret. Physics **1**, 43–55 (1946).

Bopp [Ann. Physik (5) **38**, 345–384 (1940); these Rev. **2**, 336] has proposed to eliminate infinite self-energy of the electron with a Lagrangian containing higher derivatives of the field quantities than the first. For second-order time derivatives, the field equations are of the fourth order in time-derivatives. Here such a field is quantised by a method analogous to that of Heisenberg and Pauli.

First the classical treatment is given for the Lagrangian $-\frac{1}{2}\{\sum(\partial_\alpha\psi)^2+\kappa^{-2}\sum\sum(\partial_\alpha\partial_\beta\psi)^2\}+\rho\psi$ giving the field equation $\Box(1-\kappa^{-2}\Box)\psi=-\rho$. Born's principle [Born and Infeld, Proc. Roy. Soc. London. Ser. A. **144**, 425–451 (1934)] is used to give the equation of motion of a point singularity and Born's equation is shown not to contain the effect of radiation reaction. The field equations are given in Hamiltonian form and the field quantities are quantised, the coordinates of the singularity being treated as c-numbers. The energy is the sum of the field Hamiltonian and an expression concentrated on the singularity. The field is made up of two kinds of Bose particles, one of positive, the other of negative energy. The total field energy is not positive definite and hole theory cannot, of course, be used to avoid this difficulty. The term interpreted as the self-energy of the singularity is finite but not the time-component of a four-vector. In semi-classical correspondence the energy of the singularity is that of a particle of negative mass. *C. Strachan* (Aberdeen).

Petiau, Gérard. Sur quelques propriétés des corpuscules de spin $\frac{1}{2}h/2\pi$ dans les champs électromagnétiques. J. Phys. Radium (8) **9**, 218–224 (1948).

Vrkljan, V. S. Sur le problème du proton et du neutron. J. Phys. Radium (8) **9**, 26–32 (1948).

The author proposes to represent the wave function of a proton or a neutron by two four-component spinors each of which satisfies the Dirac equation for a particle of mass M_0. Two constants n' and n'' are introduced in defining the four-dimensional current vector of each of these particles. It is shown that for a wave packet the difference of these constants enters into the expression for the convective current and their sum enters into the expression for the magnetic moment. By a proper choice of these constants agreement may be obtained with the observed values for the magnetic moments of the proton and of the neutron, respectively. *A. H. Taub* (Urbana, Ill.).

Drăganu, Mircea. Quelques remarques sur la structure du champ mésonique dans la théorie de Dirac-Proca du proton. Mathematica, Timişoara **23**, 146–152 (1948).

Manneback, C. Progrès récents de la théorie quantique des champs et du méson. Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli **17**, 26 pp. (1947).

Gião, Antonio. Théorie des particules fondamentales. II. Particules non élémentaires (protons, neutrons, mésons). Portugaliae Math. **7**, 1–44 (1948).

For part I cf. Portugaliae Math. **6**, 67–114 (1947); these Rev. **9**, 558.

Gião, Antonio. Intensité et probabilité dans les systèmes spatio-temporels. Bol. Soc. Portuguesa Mat. Sér. A. **1**, 29–40 (1947).

The author defines Laplacian and Hamiltonian systems according to whether the equations of motion are expressible in terms of differential operators of the second or first order, arguing that the first category includes the second but that the second includes the first only if relations analogous to the commutation laws of the Dirac α-matrices are valid. From a study of the conservation laws of such systems he concludes that wave mechanics of a conservative system cannot be interpreted in terms of probability phenomena in the general case where gravitational and electromagnetic forces exist. There follows a discussion of the effect of metric fields on the wave equations. *H. C. Corben.*

Gião, Antonio. Sur l'effet mécanomagnétique à l'intérieur des masses sphériques en rotation. Application au champ magnétique terrestre. C. R. Acad. Sci. Paris **226**, 645–647 (1948).

From his previous work [same C. R. **225**, 924–926 (1947); these Rev. **9**, 320] the author gives a value for the magnetic field in the interior of the earth in order to compare it with measurements of Hales and Gough [Nature **160**, 746 (1947)] in a Transvaal mine-shaft. *C. Strachan* (Aberdeen).

Gião, Antonio. Propriétés magnétiques de la matière en rotation. Gaz. Mat., Lisboa **8**, no. 34, 9–12 (1947); **9**, no. 35, 10–12 (1948).

Cf. the preceding review.

Tyablikov, S. V. A quantum-mechanical discussion of the dynamics of a crystal lattice. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 368–373 (1948). (Russian)